ABSTRACT
A large body of network-related problems can be formulated or explained by Moore-Penrose inverse of the graph Laplacian matrix of the network. This paper studies the impact of overlaying or removing a subgraph (inserting / removing a group of links, or modifying a set of link weights) on Moore-Penrose inverse of the Laplacian matrix of an existing network topology. Moreover, an iterative method is proposed to find point-to-point resistance distance and network criticality of a graph as a key performance measure to study the robustness of a network when we have link insertion and/or link removal.

1. INTRODUCTION
Moore-Penrose Inverse of Laplacian matrix plays an important role in analyzing the structural behaviour of networks. An abundance of graph theoretic metrics that are quantifying network characteristics such as connectivity, convergence speed, resiliency and robustness are defined as functions of the Moore-Penrose inverse Laplacian matrix.

Average resistance distance or network criticality is an important graph metric that can be expressed as a function of Moore-Penrose inverse Laplacian. In an electrical resistor circuit if we apply a unit current source between node \( i \) and \( j \) where \( j \) is grounded, the resistance seen between nodes \( i \) and \( j \) is called resistance distance and equals the voltage drop between \( i \) and \( j \). Resistance distance between nodes \( i \) and \( j \) in a weighted graph is defined likewise by interpreting a weighted undirected graph as a resistive network with link weights equal to the conductance of the corresponding link resistor (the difference is that in networks we don’t have the notion of unique ground). Network criticality is defined as the average of all point-to-point resistance distances between each pair of nodes in a network. It has been shown that the point-to-point and average resistance distance (or network criticality) can be written in terms of Moore-Penrose inverse of the Kirchhoff or Laplacian matrix of the network [1].

2. MOTIVATION
Average resistance distance or network criticality has a great many applications in different areas. Interested reader is referred to [2] for a sample list of such applications. Here we explain three recent applications that motivated the current research.

It has been shown that in a network of queues where each communication link is modeled as an M/G/1 queue, the round trip delay between any two nodes is upper bounded by their point-to-point resistance distance [3]. In this application we intend to insert new links to minimize the round trip delay and the question is where to add the link(s).

The second application of interest is multi-layer IP over DWDM network design. By applying Little’s formula in IP packet layer one can show that network criticality can control the onset of congestion in a network of queues [4]. On the other hand, routing process in IP layer is energy hungry [5]; therefore, one objective in this application is to find place(s) to insert links in optical transport layer to bypass some of the IP routers, so that we can control the onset of congestion and energy consumption simultaneously.

The third application of interest is about building virtual networks. Suppose we have a network of resources and some large-scale customers. Each customer asks for a subset of resources which builds the virtual network of the customer, and we need to know how we can assign virtual networks to customers to keep the resistance distance of the whole network below a certain threshold.

In this paper we focus on the effect of overlaying or deleting a subgraph; that is the insertion, removal, or modification of weighted links (one link or a group of links), on the Moore-Penrose inverse Laplacian and network criticality. The other objective is to find the change in network robustness after weighted link insertion, removal, or modification.

3. MAIN RESULT
We denote a graph by \( G_{n,m}(V,E,W) \), the graph consists of a finite node set \( V \) which contains \( n \) nodes, a set of \( m \) links denoted by \( E \in V \times V \), and a set of non-negative link weights represented by diagonal matrix \( W \), that is for link \( l = (i,j) \) (i.e. link between nodes \( i \) and \( j \)) the weight is \( w_{ij} \).

Let \( \vec{u}_{ij} \) denote a column vector in which \( i^{th} \) entry is \( 1 \), \( j^{th} \) entry is \( -1 \) and all other entries are 0, that is \( \vec{u}_{ij} \) equals the difference of \( i^{th} \) and \( j^{th} \) standard unit basis vectors.

Moreover, let \( B \) denote incidence matrix of the graph, if we label links of the graph by \( l_1, l_2, ..., l_m \), \( B \) is an \( n \times m \) matrix whose column \( k \) is equal to \( \vec{u}_{ij} \) if \( l_k \) is a link connecting...
node $i$ to node $j$. Assume $W$ is a diagonal matrix whose main diagonal represents weights of the links, and let $L$ be the Laplacian matrix of the graph, it is a well-known fact (see [4] and references therein) that:

$$L = BWB^T = \sum_{i,j} w_{ij} \tilde{u}_{ij}^l \tilde{u}_{ij}^T$$  \hspace{1cm} (1)

Moreover, if we denote resistance distance between nodes $i$ and $j$ with $\tau_{ij}$ and network criticality by $\hat{\tau}$, it has been shown ([4] and references therein) that:

$$\tau_{ij} = \frac{1}{\tilde{u}_{ij}^l L^+ \tilde{u}_{ij}^T}$$  \hspace{1cm} (2)

$$\hat{\tau} = \frac{2}{n-2} Tr(L^+)$$  \hspace{1cm} (3)

where $L^+$ denotes Moore-Penrose inverse of the Laplacian matrix. Throughout, we will also use the following facts from matrix algebra.

FACT 3.1. Let $P \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{n \times m}$, and $R \in \mathbb{R}^{m \times m}$. Assume $P$ and $R$ are positive semidefinite and positive definite matrices respectively, then:

$$(P + QRQ^T)^+ = P^+ - P^+ Q (R^{-1} + Q^+ P) Q^T P^+$$

if and only if $PP^+ Q = Q$, where $P^+$ and $Q^T$ denote the Moore-Penrose inverse of $P$ and transpose of $Q$ respectively.

PROOF. See [6]. □

FACT 3.2. Let $P \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{n \times m}$, and $R \in \mathbb{R}^{m \times m}$. Assume $P$, $P + QR$ and $I + RP^{-1} Q$ are nonsingular matrices, then:

$$(P + QR)^{-1} Q = P^{-1} Q (I + RP^{-1} Q)^{-1}$$

PROOF. See [6]. □

3.1 Moore-Penrose Inverse of Laplacian

Suppose a group of weighted links between specific points are to be installed in a network, we would like to study the impact of the new links on the Moore-Penrose Laplacian of the graph and its associated metrics, in particular average resistance distance or Network Criticality ([4]). To be more specific, Let $G_{n,m_1}$ be the graph representation of current connected network with $n$ nodes and $m_1$ links ($l_1, l_2, ..., l_{m_1}$). Assume we add links $l_{m_1+1}, l_{m_1+2}, ..., l_m$ ($m > m_1$). Note that in general the installed links are not required to be new links, if a link is already established between two nodes, adding another link between same two nodes means changing the link weight. According to equation (1) we have

$$L = BWB^T = B_1W_1B_1^T + B_2W_2B_2^T = L_1 + L_2$$  \hspace{1cm} (4)

where $W_1$ and $W_2$ denote diagonal weight matrices for links $l_1, l_2, ..., l_{m_1}$ and $l_{m_1+1}, l_{m_1+2}, ..., l_m$, respectively, and

$$L_1 = B_1W_1B_1^T = \sum_{(i,j) \in L_1} l_{ij} \tilde{u}_{ij}^l \tilde{u}_{ij}^T$$  \hspace{1cm} (5)

$$L_2 = B_2W_2B_2^T = \sum_{(i,j) \in L_2} l_{ij} \tilde{u}_{ij}^l \tilde{u}_{ij}^T$$  \hspace{1cm} (6)

Let $P = L_1$, $Q = B_2$ and $R = W_2$, then using FACT 3.1 we have:

$$L^+ = L_1^+ - L_1^+ B_2 (W_2^{-1} + B_2^T L_2 B_2)^{-1} B_2^T L_1^+$$  \hspace{1cm} (7)

Now assume $P = W_2^{-1}$, $Q = B_2^T$, $R = L_1^+ B_2$, using FACT 3.2, we get

$$(W_2^{-1} + B_2^T L_2 B_2)^{-1} B_2^T = W_2 B_2^T (I + L_1^+ B_2 W_2 B_2^T)^{-1}$$

Considering equation (6) we have:

$$(W_2^{-1} + B_2^T L_2 B_2)^{-1} B_2^T = W_2 B_2^T (I + L_1^+ L_2)^{-1}$$  \hspace{1cm} (8)

Finally, applying equation (8) in (7) results in:

$$L^+ = L_1^+ - L_1^+ L_2 (I + L_1^+ L_2)^{-1} L_1^+$$  \hspace{1cm} (9)

Equation (9) expresses the Moore-Penrose inverse of the Laplacian of new graph based on the Moore-Penrose Laplacian of the original graph (i.e. $L_1^+$) and the Laplacian matrix of the overlaid graph or the set of inserted / modified links (i.e. $L_2$).

OBSERVATION 3.3. Equation (9) holds only if $L_1 L_2^T = B_2$. Since we assumed the original graph is connected, $L_1 L_2^T$ will be a projection matrix onto the subspace of $\mathbb{R}^n$ perpendicular to $\tilde{1}$, that is $L_1 L_2^T = I - \frac{1}{n} \tilde{1} \tilde{1}^T$, where $\tilde{1}$ is an $n \times 1$ matrix with all entries equal to 1 (see [4] and references therein); hence:

$$L_1 L_2^T = (I - \frac{1}{n} \tilde{1} \tilde{1}^T) B_2 = B_2 - \frac{1}{n} J B_2 = B_2$$

Note that $J B_2 = 0$ because $B_2 = \sum \tilde{u}_{ij}^l \tilde{u}_{ij}^T$ and $J \tilde{u}_{ij}^l \tilde{u}_{ij}^T = 0, \forall (i', j') \in E$.

More generally, $L_1 L_2^T B_2$ holds if the range space of $B_2$ is a subset of range space of $L_1$. Since range space of a matrix is the space spanned by its column vectors, equation (9) is valid if adding the set of links does not increase the rank of original Laplacian matrix.

OBSERVATION 3.4. Equation (9) shows the impact of overlaying a subgraph on the Moore-Penrose Laplacian of a graph, with the same approach it is easy to derive the iterative formula for when we remove a group of links. In the following the final formula for the case of group link removal is provided:

$$L^+ = L_1^+ + L_1^+ L_2 (I - L_1^+ L_2)^{-1} L_1^+$$  \hspace{1cm} (10)

where $L_1$ and $L_2$ denote the Laplacian matrix of the original graph and the set of removed links respectively. Note that equation (10) is valid only if the rank of the Laplacian matrix of the residual network after group link removal remains the same as the original network.

An important special case is when only one link has to be installed (or removed). Suppose we would like to add a link between nodes $i$ and $j$, starting from FACT 3.1 we have:

$$L^+ = (L_1 + \tilde{u}_{ij} w_{ij} \tilde{u}_{ij}^T)^+$$

$$= L_1^+ - L_1^+ \tilde{u}_{ij} (w_{ij}^{-1} + \tilde{u}_{ij}^T L_1^+ \tilde{u}_{ij})^{-1} \tilde{u}_{ij}^T L_1^+$$

$$= L_1^+ - \frac{w_{ij}}{1 + w_{ij} \tilde{u}_{ij}^T L_1^+ \tilde{u}_{ij}} (L_1^+ \tilde{u}_{ij} (L_1^+ \tilde{u}_{ij})^T)$$  \hspace{1cm} (11)

OBSERVATION 3.5. For a case where link weights are modified (useful for virtual network building application), it can be shown that equation (9) holds as long as the weights of the residual graph are non-negative.
3.2 Point-to-Point Resistance Distance

Now we turn our attention on finding an iterative formula for point-to-point resistance distance when a link with weight $w_{ij}$ is inserted between nodes $i$ and $j$. Using equations (2) and (11) we have:

$$\hat{\tau}_{sd} = \hat{\tau}_{sd}L_t^+\bar{u}_{sd}$$
$$= \hat{\tau}_{sd}L_t^+\bar{u}_{sd}$$
$$= \frac{w_{ij}}{1 + w_{ij}(\hat{\tau})_{ij}}\frac{1}{\hat{\tau}_{sd}L_t^+\bar{u}_{ij}}L_t^+\bar{u}_{ij}L_t^+\bar{u}_{ij}L_t^+\bar{u}_{ij}L_t^+\bar{u}_{ij}L_t^+\bar{u}_{ij}\bar{u}_{sd}$$
$$= (\hat{\tau})_{sd} - \frac{w_{ij}}{1 + w_{ij}(\hat{\tau})_{ij}}(\hat{\tau}_{sd}L_t^+\bar{u}_{ij})^2$$

(12)

In particular, the point-to-point resistance distance of link between nodes $i$ and $j$ after link insertion can be iteratively found from the initial point-to-point resistance as follows (by replacing $s$ and $d$ with $i$ and $j$ in equation (12)):

$$\tau_{ij} = (\tau_{ij}) - \frac{w_{ij}}{1 + w_{ij}(\hat{\tau})_{ij}}(\hat{\tau}_{ij}L_t^+\bar{u}_{ij})^2$$
$$= (\tau_{ij}) - \frac{w_{ij}}{1 + w_{ij}(\hat{\tau})_{ij}}(\tau_{ij}L_t^+\bar{u}_{ij})$$
$$= \frac{w_{ij}}{1 + w_{ij}(\tau_{ij})}$$

(13)

Observation 3.6. Equation (13) can be re-written as follows:

$$\tau_{ij} = \frac{1}{(\tau_{ij}) + w_{ij}}$$

(14)

In other words, the point-to-point resistance distance between $i$ and $j$ after link insertion (i.e. $\tau_{ij}$) is half of the harmonic average of original point-to-point resistance distance (i.e. $(\tau_{ij})$) and the reciprocal of new inserted weight ($w_{ij}$). Equation (14) resonates with our knowledge of effective resistance concept in resistive electric circuits.

3.3 Network Criticality

Now we find an incremental formula to calculate the average resistance distance or network criticality of the whole network.

$$\dot{\hat{\tau}} = \frac{2}{n-1}Tr(L_t^+)$$
$$= \hat{\tau}_1 - \frac{2}{n-1 + \hat{\tau}_1}\frac{w_{ij}}{1 + w_{ij}}Tr((L_t^+\bar{u}_{ij})(L_t^+\bar{u}_{ij})^t)$$
$$= \hat{\tau}_1 - \frac{2}{n-1 + \hat{\tau}_1}\frac{w_{ij}}{1 + w_{ij}}Tr((L_t^+\bar{u}_{ij})(L_t^+\bar{u}_{ij}))$$
$$= \hat{\tau}_1 + \frac{w_{ij}}{1 + w_{ij}}\hat{\tau}_1 - \frac{1}{1 + w_{ij}}\frac{w_{ij}}{n-1} - 1$$
$$= \hat{\tau}_1 + \frac{w_{ij}}{1 + w_{ij}}\hat{\tau}_1 - \frac{w_{ij}}{1 + w_{ij}}\frac{w_{ij}}{n-1} - 1$$

(15)

where $(L_t^+)_i$ denotes $i^{th}$ column of $L_t^+$ and we have used the following fact: $\frac{\partial}{\partial w_{ij}} = \frac{w_{ij}}{n-1} - 1$ (see [4]).

Observation 3.7. For single link removal the steps are very similar to the link insertion as explained above, so we omit the details of this part, the final iterative formula for Moore-Penrose Laplacian and network criticality in case of link removal are as follows:

$$L^+ = L_t^+ + \frac{w_{ij}}{1 - w_{ij}^2}(L_t^+\bar{u}_{ij})(L_t^+\bar{u}_{ij})^t$$
$$\dot{\hat{\tau}} = \hat{\tau}_1 - \frac{w_{ij}}{1 - w_{ij}\hat{\tau}_1}\frac{\partial}{\partial w_{ij}}$$

While it is possible to derive iterative methods when we have both link insertion and link removal, it is preferred to use an iterative algorithmic approach based on methods explained throughout this paper. As long as the graph remains connected, the order or sequence of insertion and/or removal of the links won’t change the result.

4. CONCLUSIONS AND ROADMAP

Application of graph theoretic techniques based on analysis of the graph Laplacian matrix in communication networks has witnessed rapid growing in recent years, but there is a gap between theoretical concepts and their corresponding feasible deployment algorithms. In order to bridge this gap the first step is to introduce appropriate methods to calculate Moore-Penrose inverse of the Laplacian matrix and related graph metrics incrementally. In this paper we introduced one such incremental method and proposed and iterative approach to find Moore-Penrose inverse Laplacian when we experience link insertion and/or link removal, then we used the result to derive an iterative method to find average resistance distance.

Our work in this paper was mainly based on the assumption that removal/insertion of links does not change the cardinality of the graph Laplacian. We are currently finalizing another phase of our research in which we have relaxed this constraint, clearly the iterative methods in general case cannot have simple forms anymore since FACT 3.1 does not hold and we cannot use it.

In a related work, we are designing iterative algorithms to dynamically configure the network in a data center to match connectivity to the traffic using the concept of point-to-point resistance distance.

5. REFERENCES