1. Introduction

We consider a stochastic capacity problem of dynamically matching the supply of resources and the uncertain demand for resources over a planning horizon of $T$ periods. If demand exceeds the resource capacity in any time period, then excess demand is lost and a penalty is incurred for the amount of lost demand. The discrete-time model reflects the relative time granularity at which resource capacity decisions can be made, whereas a lead time $L > 0$ represents in the model the relative time-scale difference between when a resource capacity decision is made and when this capacity decision takes effect. Our optimal control problem then is to determine the best policy for making resource capacity allocation decisions in each period, based on the demand realized up to that period, with the goal of maximizing net-benefit in expectation over the planning horizon.

The net-benefit structure of each period in our model consists of a per-unit penalty for insufficient resource capacity to serve uncertain demand and a per-unit cost for the amount of resource capacity allocated. Such resource capacity problems arise within many different computing/networking applications, including capacity provisioning for cloud computing environments where high-frequency demand, having considerable uncertainty, needs to be served by resource capacity allocations over planning horizons during which any decisions to change the resource capacity allocation requires relatively long lead times. This resource capacity allocation model can be mapped in a mathematically equivalent manner to lost-sales inventory models [6]. In particular, the optimal order policy of the corresponding stochastic inventory control problem can be directly mapped to the optimal resource capacity policy, where the capacity allocation for any period is equivalent to the (nonnegative) inventory at the end of the period. We therefore consider henceforth our stochastic capacity planning problem within the context of stochastic lost-sales inventory control problems.

Lost-sales inventory models have a long and rich history in the research literature, starting with their introduction by Bellman et al. [1]. Certain properties of the optimal policy have been investigated by various researchers; see, e.g., [8]. With respect to computation of the optimal policy, the primary approach has been dynamic programming, combined with various heuristics to speed up computations [7]. However, since the state-space of any such dynamic program grows exponentially in the lead time, such computations become extremely challenging even for lead times less than ten [7]. This difficulty has led to a considerable body of work on approximations, including online algorithms that have provable performance guarantees [4] and approximate dynamic programming algorithms [2]. The work closest to our own is that of Reiman [5], who studies a very simple open-loop constant-order policy under a lost-sal es model, with positive lead times and demand arriving as a Poisson process, in comparison with the best base-stock policy. However, Reiman makes no attempt to compare either policy to the true optimal policy, which he notes is unknown.

Herein we derive theoretical results that address the open question regarding the relationship between a constant-order policy and the optimal policy for stochastic lost-sales inventory control problems with large lead times, which in turn establish theoretical results to address the corresponding question within the context of the class of stochastic capacity planning problems motivating the present study. Specifically, we prove that, as the lead time grows large, the best constant-order inventory control policy is in fact asymptotically optimal. We also establish explicit bounds on how large the lead time should be to ensure that the best constant-order policy incurs an expected cost of at most $1 + \epsilon$ times that incurred by the optimal policy, for any $\epsilon > 0$, which to the best of our knowledge is the first result of this kind for lost-sales models when the lead time is large and the runtime does not grow with the lead time. Our main proof technique combines a novel coupling for suprema of random walks with arguments from queueing theory.

Section 2 formally defines the model and formulation, and Section 3 states our main results. We sketch the proof of our main results in Section 4, including additional results of interest. The reader is referred to [3] for a complete statement of results, proofs, related work, and technical details.

2. Model and Formulation

Let us consider a standard lost-sales inventory optimal control problem. One is given as input the holding cost, lost-demand penalty, time horizon, lead time, and demand distribution. The problem then is to determine the optimal ordering policy to control inventory in this so-called single-item, discrete-time, periodic-review, lost-sales model.

Specifically, we consider the following lost-sales model and associated optimal control problem. Let $h$ denote the per-unit holding cost and $c$ the per-unit lost-demand penalty. Time is slotted, with the time horizon and lead time comprised of $T$ and $L$ periods, respectively, where $T > L >
0. The demand in each period is assumed to be independent and identically distributed (i.i.d.) according to a non-negative demand distribution $D$ with finite second moment. At the start of each time period $t$ there is an amount of inventory $I_t$ available. There is also an $L$-dimensional pipeline $x_t = (x_{t1}, x_{t2}, \ldots, x_{tL})$ that represents the vector of orders placed before period $t$, but not yet received. The system dynamics for period $t$ then proceed as follows. First, a new amount $x_{t1}$ of goods is added to the inventory. Second, before seeing the demand of period $t$, an order for more inventory is placed. This order must be a (possibly random) function only of the time horizon $T$, the current time $t$, the inventory level at the start of period $t$ ($I_t$), the pipeline vector at the start of period $t$ ($x_t$), and the model primitives $L, h, c, D$. In particular, the ordering decision at time $t$ cannot depend on the realizations of future demand. We call all such policies admissible policies, and denote the family of admissible policies by $\Pi$.

The pipeline vector is subsequently updated in a manner analogous to that of a queue: the front entry $x_{t1}$ is removed, all other entries move up one position, and the new order is appended at the end ($x_{tL+1}$). Next, a random demand $D_t$ is drawn independently from $D$. The inventory is then updated according to $I_{t+1} = (I_t + x_{t1} - D_t)^+$ where $y^+ = \max\{y, 0\}$, noting that $D_t$ is independent of $I_t + x_{t1}$. Of course, some demand in the period may be lost. Specifically, the amount of demand lost (due to insufficient inventory on hand) in period $t$ is denoted by $N_t = (I_t + x_{t1} - D_t)^-$ where $y^- = \min\{y, 0\}$. At the end of period $t$ (but before the start of period $t+1$), the inventory incurs a holding cost (for excess inventory) equal to $hI_{t+1}$ and a lost-sales penalty (for lost demand) equal to $cN_t$.

The goal of the inventory control problem is to minimize the expected cost incurred over the entire time horizon. In particular, suppose $I_0 = 0$ and $x_0 = 0$, let us define

$$C_t \triangleq hI_{t+1} + cN_t = h(I_t + x_{t1} - D_t)^+ + c(I_t + x_{t1} - D_t)^-.$$  

Then we seek to find the policy $\pi \in \Pi$ that minimizes $E[\sum_{t=1}^{T} C_t]$, where the expectation is over the random demand and any random decisions taken by policy $\pi$. For a given policy $\pi$, let $N^\pi, C^\pi, I^\pi, x^\pi_t; t = 1, \ldots, T$ denote the associated random variables (r.v.s) when policy $\pi$ is implemented (all constructed on the same probability space). The corresponding lost-sales inventory control problem is thus given by $\inf_{\pi \in \Pi} E[\sum_{t=1}^{T} C^\pi_t]$, or equivalently

$$\inf_{\pi \in \Pi} \sum_{t=1}^{T} E[h(I^\pi_t + x^\pi_t - D_t)^+ + c(I^\pi_t + x^\pi_t - D_t)^-].$$  

3. Main Results

Let $D$ denote a realization from $D$. Note that if the same deterministic quantity $r < E[D]$ is ordered in every period, then the inventory evolves exactly as the waiting time in a $GI/D/1$ queue (initially empty) with interarrival time distribution $D$ and deterministic processing time (the constant) $r$. Let $I^\pi_\infty$ denote a r.v. distributed as the steady-state waiting time in the corresponding $GI/D/1$ queue; namely, $I^\pi_\infty \sim \text{sup}_t \{kr - \sum_{i=1}^{t} D_i\}$. Lastly, we define several functions that will be instrumental for our analysis:

$$z \triangleq \arg\min_{v \geq 0} \left( hE[I^\infty_\infty] - cv \right);$$

$$f \triangleq \max_{v > 0} \left( 1 - \frac{cE[D]}{hE[(r-D)^+]} \right) E[(D-v)^+];$$

$$g \triangleq \min_{v \geq 0} hE[(D-v)^+] + cE[(D-v)^+];$$

$$y(\epsilon) \triangleq \max \left( 2000(1 + f^{-1} \epsilon) (1 + E[D]^2 + E[D]^2)^4 \times \left( 1 + (1 + h^{-1})c + (1 - c^{-1})h \right)^3 (1 + g^{-1}) \epsilon^{-1}, 300c^3 E[D]^2 h^{-2} \epsilon^{-2} \right).$$

We will later show that $z$ is the best constant possible if the same constant amount has to be ordered in every time period. Note that $z \in [0, E[D])$, since $E[I^\infty_\infty] = 0$ and $\lim_{t \rightarrow \infty} E[I^\infty_\infty] = \infty$.

We now formally state our main results. For $r \in \mathbb{R}^+$, let $\pi_r$ denote the policy that orders the sum of the r.v. $I^\pi_\infty$ and the deterministic quantity $r$ (i.e., $I^\pi_\infty + r$) in the first time period, and then orders $r$ in all subsequent time periods. Let $OPT$ denote the optimal value of the lost-sales inventory control problem (1) for a given $L$ and $T$, or an appropriately defined lim sup over $T$ if the optimal value is not actually attained. We then have our main theorem and an important corollary as follows.

**Theorem.** For all $\epsilon \in (0, 1), L \geq y(\epsilon),$ and $T \geq 2cE[D]g^{-1}\epsilon^{-1}L$, 

$$E[\sum_{t=1}^{T} C^\pi_t] \leq OPT \leq E[\sum_{t=1}^{T} C^\pi_t].$$

**Corollary.** 

$$\lim_{L \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{E[\sum_{t=1}^{T} C^\pi_t]}{OPT} = 1.$$ 

In particular, the simple constant-order policy is asymptotically optimal as $L \rightarrow \infty$. Although the explicit bounds given by $y(\epsilon)$ could be improved by a more careful analysis, we believe that the dependence on the parameters $f^{-1}, E[D]^2$ and $ch^{-1}$ is fundamental, and leave a tighter bound analysis as an interesting open question.

4. Proof Sketch

As a first step towards proving our main results, in the next two lemmas, we explicitly characterize the cost incurred by any policy $\pi \in \Pi$ during any consecutive $L$ time periods, and we derive a lower bound on this cost. For positive integers $j, k$, let $\delta_{j,k}$ equal 1 if $j = k$, and 0 otherwise. Then we conclude the following, from the fundamental dynamics of the inventory process.

**Lemma 3.** For any policy $\pi \in \Pi$ and time $\tau \in [1, T-L],$

$$E \left[ \sum_{t=\tau+1}^{\tau+L-1} C^\pi_t \right] = h \sum_{k=1}^{L} \max_{j=0, \ldots, k} \left( \sum_{i=k+1}^{k} (x^\pi_i - D_{\tau+i-1}) + \delta_{j,k} I^\pi_\infty \right) + c \left( E[I^\pi_\infty x^\pi_t, I^\pi_t] - I^\pi_t + LE[D] - \sum_{i=1}^{L} x^\pi_i \right).$$
Let \((x^*, I^*)\) denote any solution to the optimization problem

\[
\min_{x \in \mathbb{R}^+, I \in \mathbb{R}^+} E \left[ \sum_{t=1}^{r+L-1} C_t | x_t = x, I_t = I \right],
\]

where the existence of \((x^*, I^*)\) follows from a routine continuity and compactness argument. Note that, intuitively, \((x^*, I^*)\) corresponds to the “best” pipeline-inventory state that any policy could possibly see, so as to minimize the expected cost incurred over the next \(L\) periods. To deal with potential problems “at the boundary”, it will be useful to fix some integer \(L' \in [1, L]\) and define \(x^{r'}\) to be the vector whose first \(L'\) components are identical to those of \(x^*\), but whose final \(L - L'\) components are all set to zero. As a notational convenience, we also define \(I^{r'} = I^*\). In that case, again using the inventory dynamics and straightforward algebraic manipulations, we conclude the following.

**Lemma 4.** For any \(\pi \in \Pi\) and \(r \in [1, T-L]\),

\[
E \left[ \sum_{t=r+1}^{T-1} C_t \right] \geq h \sum_{k=1}^{L} \max_{i=0, \ldots, k} \left( \sum_{i=k+1}^{L} x_t^{r'} - \sum_{i=1}^{j} D_i + \delta_{i,k} I^{r'} \right) + E[I^*_{L+1} - I^{r'} + LE[D] - \sum_{i=1}^{L} x_t^{r'} - E[D](L - L')] + cE[D](L - L').
\]

The next step towards establishing our main results is based on an analysis of the dynamics of the constant-order policy. Namely, we explicitly characterize the cost incurred by the policy \(\pi_t\). As previously noted, if the same deterministic quantity \(r \) is ordered in every period, the inventory evolves exactly as the waiting time in a \(GI/D/1\) queue (initially empty) with interarrival distribution \(D\) and deterministic processing time \(r\). It follows that \(\{I^*_{L+k}; k \geq 2\} \) is a stationary sequence of r.v.s, with \(I^*_{L+k}\) distributed as \(I^*_L\) for all \(k \geq 2\). We then derive an expression for the cost incurred by the constant-order policy \(\pi_t\) in terms of the r.v. \(I^*_L\), from which it follows that \(z\) is indeed the best constant among all constant-order policies:

**Lemma 5.** For any \(r \in [0, E[D])\) and \(t \geq L + 1\),

\[
E[C^*_t] = hE[I^*_L] + c(E[D] - r).
\]

**Corollary 6.** For any \(r \in [0, E[D])\),

\[
E\left[ \sum_{t=1}^{T} C^*_t \right] \geq E\left[ \sum_{t=1}^{T} C_t \right].
\]

The final step in the proof of our main results consists of characterizing the difference in performance of the constant-order policy and the lower bound. To accomplish this, we do not consider the aforementioned “optimal” constant-order policy. Instead, we consider a constant-order policy which tries to “mimic” the inventory dynamics if the initial pipeline-inventory vector equals \((x^*, I^*)\). Specifically, we derive an upper bound on the difference between the expected cost of a particular constant-order policy and that of a general policy \(\pi\). To this end, let us define \(r^{*'} \triangleq L^{-1} \sum_{t=1}^{L} x_t^*\). Also, supposing \(r^{*'} < E[D]\), let \(t^{*'}_\infty\) denote the random (largest) index at which the negatively-drifted random walk (initially empty) with i.i.d. step-size \(r^{*'} - D\) attains its supremum. Through a careful coupling between the inventory dynamics under policy \(\pi_{r^{*'}}\) and the inventory dynamics over the first \(L\) time periods if the initial pipeline-inventory vector equals \((x^*, I^*)\), combined with several bounds for the suprema of the associated random walks, we establish the following bound on the performance difference between the constant-order policy and any other policy.

**Theorem 7.** If \(r^{*'} < E[D]\), then for any \(\pi \in \Pi\) and \(r \in [L + 1, T-L]\),

\[
E[\sum_{t=r+1}^{T-1} C_t] \leq hLr^{*'} \sum_{j=(L-L')}^{\infty} \mathbb{P}(t^{*'}_\infty \geq j) + h \sum_{k=1}^{L} \mathbb{E}[\delta_{i,k} I^{*'}_L] + cE[D](L - L').
\]

Our main result then follows by combining two types of arguments. First, we develop several worst-case bounds for various quantities appearing in and related to Theorem 7. For example, we prove that for all \(L' \leq L - \left(\lfloor (2ch^{-1})L \rfloor + 2\right)\), one has \(E[I^*_L] \leq \left(\lfloor (2ch^{-1})L \rfloor + 2\right)E[D]\). Indeed, if this were not the case, it would imply that any policy incurs a holding cost over any \(L\) periods which is strictly greater than \(cE[D]\), yielding a contradiction, since this is exactly the cost incurred by the policy which orders nothing. Next, combining several such arguments with a straightforward asymptotic analysis completes the proof of our main result. This in turn provides analogous theoretical results on the asymptotic optimality of a corresponding constant-capacity allocation policy for the class of stochastic capacity planning problems motivating our present study.

### 5. References


