ABSTRACT
A primary interest of this paper is to find tail asymptotics of the stationary distribution of a generalized Jackson network with two nodes under phase-type setting, provided its stability holds. Here, the phase type setting is meant that the arrival processes are the so called Markov arrival processes, and the service time distributions are of phase type. We consider two types of the tail asymptotics, the tail decay rate of the marginal stationary distribution in an arbitrary direction and those for the joint stationary probabilities in the coordinate directions.

There are two major reasons why those tail asymptotics are interesting particularly for the two-node generalized Jackson network. Before discussing them, we recall what is the generalized Jackson network and how it has been studied.

A queueing network in which at each node customers finishing service are independently routed according to given probabilities and their service times are i.i.d is called a generalized Jackson network when exogenous arrivals and service times at each node are independent but their distributions are general. Here, service discipline at each node is assumed to be first-come and first-served. If the exogenous arrival processes are time-homogeneous Poisson and the service times are exponentially distributed, then this network becomes the well known Jackson network. Thus, the generalized Jackson network is a natural generalization of the Jackson network. In this paper, we assume that each node has a single server.

The Jackson network has been widely used in applications because it is analytically tractable, particularly, it has a product form stationary distribution. However, this is not the case for the generalized Jackson network. This fact motivated Harrison and Williams [11] to study diffusion approximation for such networks in heavy traffic. They derived the prototype of a semimartingale reflecting Brownian motion on a nonnegative orthant, SRBM for short. Since then SRBM has been extensively studied, which became a big research area (see [4]). Furthermore, its relation to the generalized Jackson network is still a hot topic (e.g., see [3, 9]).

We now return to the reasons for our study. When a stationary distribution is difficult to analyze, it is natural to study its characteristics which are tractable in analysis but still important in application. Tail asymptotic is one of such characteristics, and has been actively studied for many years. However, we still do not have a satisfactory answer even for the two-node generalized Jackson network with single servers. This is one reason why we study its tail asymptotic problem.

Another reason is related to the two-dimensional SRBM which is obtained as a diffusion limit of a sequence of the two-node generalized Jackson networks (see [9] for its verification concerning stationary distribution). We now have good view on the tail asymptotics of the stationary distribution of a two-dimensional SRBM (e.g., see [1, 5, 6]). Hence, if we can get those for the two-node generalized Jackson network, then we can argue the quality of the diffusion approximation by them. In this consideration as well as applications of the generalized Jackson network itself, it is desirable to get the tail asymptotics in analytically tractable form.

To facilitate our analysis, we assume the phase-type setting. This certainly limits the class of queueing network models, but we believe it is sufficiently large since phase-type distributions are dense in the set of distributions on the nonnegative real line, and Markovian arrival processes are versatile. For example, the exogenous arrival processes are not necessarily to be renewal processes, which are often assumed for the generalized Jackson network (e.g., see [9]).

We consider the two-node generalized Jackson network under phase-type setting as a special case of a Markov modulated two-dimensional reflecting random walk, and study the tail asymptotic problem for this reflecting process. Thus, we answer the problem for a more general model.

Recently, Ozawa [18] introduced a two-dimensional quasi-birth-and-death process, 2d-QBD process for short, for queueing networks including the generalized Jackson network, and studied the tail asymptotics of its stationary distribution. We will take this 2d-QBD process for considering the tail asymptotic problem. However, we cannot use Ozawa’s [18] results because his answers require to numerically solve certain matrix equations. Thus, the tail asymptotics obtained in [18] are analytically intractable. Furthermore, the asymptotics are only considered for the coordinate directions.

Thus, we need to answer the problem in a different way from Ozawa’s [18]. To this end, we will show that a crucial step is to find the convergence parameter of a nonnegative infinite dimensional matrix with QBD like block matrices. This problem can be reduced to the existence of a nonnegative right invariant vector of such a matrix, and we refer to this vector as a super-harmonic vector. Thus, the problem is to find a necessary and sufficient condition in terms of block matrices for the existence of the super-harmonic
Once this convergence parameter problem is resolved, we are ready to extend all the techniques recently used to study a two-dimensional reflecting random walk with unbounded jumps in [15] for that under Markov modulation. This can be done. In this way, we obtain the tail asymptotics of the stationary distribution of the two-node generalized Jackson network. Namely, let $Z \equiv (L_1, L_2, J)$ be a random vector subject to the stationary distribution of the numbers of customers $L_1$ and $L_2$ at two nodes and the background state $J$ of the Markov chain which describes the phases of arrival streams and progress of service. Then, for any two-dimensional non-zero vector $e \geq 0$, the following limit exists and is obtained in terms of the modeling parameters.

$$\frac{1}{x} \log P((c, L) > x) \quad (x \to \infty),$$

where $(a, b)$ is the standard inner product of vectors $a, b \in \mathbb{R}^2$. We refers to this decay rate as that of the marginal stationary distribution in diction $c$. Similarly, we obtain the following limit for each fixed $\ell$ and $k$.

$$\frac{1}{n} \log P(L_i = n, L_{3-i} = \ell, J = k) \quad (n \to \infty).$$

These decay rates are referred to as those of the stationary probabilities on the coordinates axes. Their analytical expressions will be obtained in Theorem 2 below.

The tail asymptotic problem for the two-node generalized Jackson network under phase-type setting has been studied by Takahashi and his colleagues [12, 13], but they only derive upper bounds for the tail decay rates of the stationary probabilities. There are many studies for the tandem queue case (e.g., see [2, 10, 8, 19]). However, those studies are also incomplete because either extra conditions are required or the tail asymptotics are only obtained for the marginal stationary distribution of one node. By contrast, we derive the tail decay rate for the marginal distribution in an arbitrary direction, and do not require any extra condition except for the phase-type setting.

In what follows, we summarize our main results. We first introduce notations for specifying the two-node Jackson network under phase-type setting. We number the two node as 1 and 2, and assume the following dynamics.

(a) A customer which completes service at node $i$ goes to node $j$ with probability $r_{ij}$ for $i = 1, 2, j = 0, 1, 2$, where node 0 means the outside of the network, $r_{12} > 0$ and $r_{21} > 0$ and $r_{12}r_{21} < 1$. We assume that $r_{ii} = 0$ for simplicity.

(b) Exogenous customers arrive at node $i$ subject to the Markovian arrival process with generator $T_i + U_i$, and generating arrivals by rate matrix $U_i$. Here, $T_i$ and $U_i$ are finite square matrices of the same size for each $i = 1, 2$. Let $\pi_i$ be the stationary distribution of $T_i + U_i$, then the mean exogenous arrival rate at node $i$ is given by $\lambda_i \equiv (\pi_i, U_i)$, where 1 is the column vector whose entries are all units.

(c) Node $i$ has a single server, whose service times are independently and identically distributed subject to a phase type distribution with $(\beta_i, S_i)$, where $\beta_i$ is the row vector representing the initial phase distribution and $S_i$ is a transition rate matrix for internal state transitions. Here, $S_i$ is a finite square matrix, and $\beta_i$ has the same dimension as that of $S_i$ for each $i = 1, 2$. The mean service rate at node $i$ is given by $\mu_i = (\beta_i, S_i)^{-1}$, $\beta_i > 0$.

The stability condition for the existence of the stationary distribution is obvious for the generalized Jackson network. Namely, the network is stable if and only if

$$\rho_i \equiv \lambda_i + \lambda_{(3-i)}r_{(3-i)} < 1, \quad i = 1, 2. \quad (3)$$

We assume this condition throughout the paper.

For $i = 1, 2$ and $\theta = (\theta_1, \theta_2)^T$, we refers to this decay rate as that of the marginal distribution at node $i$, that is, there are positive column vectors $h^{(ua)}(\theta_i), h^{(md)}(\theta_i)$ such that

$$(T_i + e^{\theta_i}U_i)h^{(ua)}(\theta_i) = \gamma^{(ua)}(\theta_i)h^{(ua)}(\theta_i),$$

$$(S_i + t_i(\theta)D_i)h^{(md)}(\theta_i) = \gamma^{(md)}(\theta_i)h^{(md)}(\theta_i), \quad i = 1, 2.$$

Thus, $\gamma^{(ua)}(\theta_i)$ and $\gamma^{(md)}(\theta_i)$ are cumulant generating functions in the theory of large deviations (e.g., see [7]).

We are now ready to present main results. For them, the following notations are convenient.

$$\partial \Gamma = \{\theta \in \mathbb{R}^2; \gamma^{(ua)}(\theta_1) + \gamma^{(md)}(\theta_2) + \gamma^{(md)}(\theta_1) + \gamma^{(md)}(\theta_2) = 0\},$$

$$\Gamma_{max} = \{\theta \in \mathbb{R}^2; \theta' \in \partial \Gamma, \theta < \theta'\}.$$

Thus, $\gamma^{(ua)}(\theta_i)$ and $\gamma^{(md)}(\theta_i)$ are cumulant generating functions in the theory of large deviations (e.g., see [7]).

Remark 1. If we define $\Gamma$ as

$$\Gamma = \{\theta \in \mathbb{R}^2; \gamma^{(ua)}(\theta_1) + \gamma^{(md)}(\theta_2) + \gamma^{(md)}(\theta_1) + \gamma^{(md)}(\theta_2) \leq 0\},$$

then it can be shown that $\Gamma$ is a convex set. Hence, $\partial \Gamma$ can be considered as a convex curve. In the decay rate papers [14, 15, 16] for two-dimensional random walks, convex sets corresponding to $\Gamma$ are used rather than convex curves corresponding to $\partial \Gamma$. There is a technical reason why $\partial \Gamma$ is used rather than $\Gamma$, which is discussed in [17].
Let $\varphi(\theta) \equiv \mathbb{E}(e^{\varphi X})$ be the moment generating function of $X$ subject to the marginal stationary distribution with respect to the numbers of customers at nodes 1 and 2, and define $D$ as

$$D = \text{the interior of } \{ \theta \in \mathbb{R}^2; \varphi(\theta) < \infty \},$$

which is referred to as a convergence domain of $\varphi$. Denote the two extreme points of $\Gamma_i$ by

$$\theta_i^{(i,\Gamma)} = \arg\sup_{\theta \in \mathbb{R}^2} \{ \theta_i \geq 0; \theta \in \partial \Gamma_i \}, \quad i = 1, 2.$$

Using these points, we define the vector $\tau$ by

$$\tau_1 = \sup\{ \theta_1 \in \mathbb{R}; \theta \in \partial \Gamma_1; \theta_2 < \theta_2^{(2,\Gamma)} \},$$

$$\tau_2 = \sup\{ \theta_2 \in \mathbb{R}; \theta \in \partial \Gamma_2; \theta_1 < \theta_1^{(1,\Gamma)} \}.$$

**Theorem 1.** Under the stability condition,

$$D = \{ \theta \in \Gamma_{\max}; \theta < \tau \}. \quad (7)$$

**Theorem 2.** Under the same assumptions of Theorem 1,

$$\lim_{x \to -\infty} \frac{1}{x} \log P(c, L) > x = -\sup\{ u > 0; uc \in D \}, \quad (8)$$

and, for each fixed nonnegative integer $\ell$ and $j \in \mathcal{V}_{\max(1,\ell)}(\theta)$,

$$\lim_{n \to -\infty} \frac{1}{n} \log L_i = n, L_{3-i} = \ell, J = j = -\sup\{ \theta_i > 0; \theta \in D \} = -\tau_j, \quad (9)$$

where $\mathcal{V}_0(\mathcal{K}(\theta))$ is the set of background states for $L_i \geq 1$ and $L_{3-i} = 0$ ($L_{3-i} \geq 1$, respectively).

Remark that the convergence domain and the decay rates are determined by $\gamma^{(\infty)}(\theta_i)$ and $\gamma^{(1d)}(\theta_i)$, and therefore (5) and (6) suggest that Theorems 1 and 2 hold beyond the phase-type setting under the assumption that the total number of exogenous arriving customers in each finite time interval and the service times have light tail distributions, that is, their tails decay exponentially fast.

It is also notable that these decay rates are obtained in the exactly same way as those of the stationary distributions of the two-dimensional SRBM (see Theorems 2.1, 2.2 and 2.3 of [5]). However, there may be some gaps between the limit of the tail decay rates for the two-node generalized Jackson network under diffusion scaling and those of the corresponding SRBM. We are now considering this problem, and this paper is preliminary work for it.

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**1. REFERENCES**


