ABSTRACT
This paper describes the structure of optimal policies for discounted periodic-review single-commodity total-cost inventory control problems with fixed ordering costs for finite and infinite horizons. There are known conditions in the literature for optimality of \((s_t, S_t)\) policies for finite-horizon problems and the optimality of \((s, S)\) policies for infinite-horizon problems. The results of this paper cover the situation, when such assumption may not hold. This paper describes a parameter, which, together with the value of the discount factor and the horizon length, defines the structure of an optimal policy. For the finite horizon, depending on the values of this parameter and the discount factor, an optimal policy either is an \((s, S)\) policy or never orders inventory. For a finite horizon, depending on the values of this parameter, the discount factor, and the horizon length, there are three possible structures of an optimal policy: (i) it is an \((s_t, S_t)\) policy, (ii) it is an \((s_t, S_t)\) policy at earlier stages and then does not order inventory, or (iii) it never orders inventory. The paper also establishes continuity of the optimal value function and describes the optimal actions at states \(s_t\) and \(s\).

Keywords
inventory control; finite horizon, infinite horizon; optimal policy, \((s, S)\) policy.

1. INTRODUCTION
It is well-known that for the classic stochastic periodic-review single-commodity inventory control problems with fixed ordering costs, \((s, S)\) policies are optimal under certain conditions. Systematic studies of inventory control problems started with the work of Arrow et al. [1] and Dvoretzky et al. [7]. Most of the earlier results are surveyed in the books by Porteus [12]. Here, we mention just a few directly relevant references.

Scarf [13] proved the optimality of \((s, S)\) policies for finite-horizon problems with continuous demand; see Zabel [16] for corrections to [13]. Iglehart [11] extended the results in Scarf [13] to infinite-horizon problems. Veinott and Wagner [15] proved the optimality of \((s, S)\) policies for both finite-horizon and infinite-horizon problems with discrete demand. Beyer and Sethi [4] completed the missing proofs in Iglehart [11] and Veinott and Wagner [15]. In general, \((s, S)\) policies may not be optimal. To ensure the optimality of \((s, S)\) policies, the additional assumption on backordering cost function (see Condition 3.3 below) was introduced by Veinott and Wagner [15], and this or relevant assumptions are usually used in the literature; see e.g., Bensoussan [2, Theorem 9.11], Bertsekas [3] Chen and Simchi-Levi [5, 6], and Heyman and Sobel [10]. As shown by Veinott and Wagner [15] for problems with discrete demand and Feinberg and Lewis [9] for an arbitrary distributed demand, such assumptions are not needed for an infinite-horizon problem, when the discount factor is close to 1.

In order to describe the optimal actions at state \(s\) for an \((s, S)\) policy, the continuity of the value functions is needed. Simchi-Levi et al. [14, Theorem 8.3.4] and Bensoussan [2, Theorem 9.11] proved that finite-horizon and infinite-horizon value functions are continuous for inventory models with linear holding costs and continuous demand.

This paper studies the structure of optimal policies without the assumption on backordering costs mentioned above. We describe a parameter, which, together with the value of the discount factor and the horizon length, defines the structure of an optimal policy. The paper also proves the continuity of value functions. The continuity of the value functions is then used to describe the sets of optimal actions for all the states.

The rest of the paper is organized in the following way. Section 2 introduces the classic stochastic periodic-review single-commodity inventory control problems with fixed ordering costs. Section 3 presents the known results on the optimality of \((s, S)\) policies. Section 4 describes the structure of optimal policies for finite-horizon and infinite-horizon problems for all possible values of discount factors. Section 5 establishes continuity of value functions and describes the optimal actions at states \(s_t\) and \(s\).

2. MODEL DEFINITION
Let \(\mathbb{R}\) denote the real line, \(\mathbb{Z}\) denote the set of all integers, \(\mathbb{R}_+ := [0, +\infty)\) and \(\mathbb{N}_0 = \{0, 1, 2, \ldots\}\). Consider the classic stochastic periodic-review inventory control problem with fixed ordering cost and general demand. At times \(t = 0, 1, \ldots\), a decision-maker views the current inventory of a single commodity and makes an ordering decision. Assuming zero lead times, the products are immediately available to meet demand. Demand is then realized and the unmet demand is backlogged. The decision-maker views the remaining inventory, and the process continues. The state and action spaces are \(X = \mathbb{R}\) and \(A = \mathbb{R}_+\). The inventory control problem is defined by the following parameters.
1. $K > 0$ is a fixed ordering cost;
2. $c > 0$ is the per unit ordering cost;
3. $h(\cdot)$ is the holding/backordering cost per period, which is assumed to be a convex function on $\mathbb{X}$ with real values and $h(x) \to \infty$ as $|x| \to \infty$, without loss of generality, consider $h(\cdot)$ to be non-negative;
4. $\{D_t, t = 1, 2, \ldots\}$ is a sequence of i.i.d. non-negative finite random variables representing the demand at periods 0, 1, \ldots. We assume that $\mathbb{E}[h(x - D)] < +\infty$ for all $x \in \mathbb{X}$ and $\mathbb{P}(D > 0) > 0$, where $D$ is a random variable with the same distribution as $D_t$;
5. $\alpha \geq 0$ is the discount factor for finite-horizon problems and $\alpha \in (0, 1)$ for infinite-horizon problems.

Define $S_N := 0$ and $S_t := \sum_{j=1}^t D_j$, $t = 1, 2, \ldots$. Let the limit

$$k_h := \lim_{x \to +\infty} \frac{h(x)}{x} \in (0, +\infty].$$

The dynamic of the system is defined by

$$x_{t+1} = x_t + a_t - D_{t+1}, \quad t = 0, 1, 2, \ldots,$$

where $x_t$ and $a_t$ denote the current inventory level and the ordered amount at period $t$ respectively. Then the cost function at period $t$ is defined for all $(x_t, a_t) \in \mathbb{X} \times \mathbb{A}$

$$c(x_t, a_t) = K_1(x_t, a_t) + c1 + \mathbb{E}[h(x_t + a_t - D)].$$

(2.2)

For a finite horizon $N = 0, 1, \ldots$ and a discount factor $\alpha \geq 0$, define the expected total discounted cost

$$v_{N, \alpha}^*(x) := \mathbb{E}^* \left[ \sum_{t=0}^{N-1} \alpha^t c(x_t, a_t) \right].$$

(2.3)

When $N = +\infty$ and $\alpha \in (0, 1)$, (2.3) defines the infinite horizon expected total discounted cost of $\pi$ denoted by $v_{\infty, \alpha}^*(x)$ instead of $v_{N, \alpha}^*(x)$. Define the optimal cost $v_{N, \alpha}(x) = \inf_{\pi \in \Pi} v_{N, \alpha}^*(x)$, and $v_\alpha(x) = \inf_{\pi \in \Pi} v_{\infty, \alpha}^*(x)$, where $\Pi$ is the set of all policies. A policy $\pi$ is called optimal for the respective criterion if $v_{N, \alpha}(x) = v_{N, \alpha}^*(x)$ or $v_\alpha(x) = v_\alpha^*(x)$ for all $x \in \mathbb{X}$.

### 3. OPTIMALITY OF $(s, S)$ POLICIES

This section considers known sufficient condition for the optimality of $(s_1, S_1, \alpha)$ and $(s_\alpha, S_\alpha, \alpha)$ policies for discounted problems. The value functions for the inventory control problem defined in Section 2 can be written as

$$v_{t+1, \alpha}(x) = \min_{a \geq 0} \{KI_{\alpha > 0} + G_{t, \alpha}(x + a)\} - ce, \quad t = 0, 1, \ldots,$$

(3.1)

$$v_\alpha(x) = \min_{a \geq 0} \{KI_{\alpha > 0} + G_\alpha(x + a)\} - ce,$$

(3.2)

where $I_B(x)$ is an indicator function and

$$G_{t, \alpha}(x) = cx + \mathbb{E}[h(x - D)] + \alpha \mathbb{E}[v_{t, \alpha}(x - D)], \quad t = 0, 1, \ldots,$$

(3.3)

and

$$G_\alpha(x) = cx + \mathbb{E}[h(x - D)] + \alpha \mathbb{E}[v_\alpha(x - D)],$$

(3.4)

and $v_{0, \alpha}(x) = 0$ for all $x \in \mathbb{X}, \alpha \geq 0$ in equalities (3.1), (3.3), and $\alpha \in [0, 1]$ in equalities (3.2), (3.4).

Recall the definitions of $K$-convex functions and $(s, S)$ policies.

### 3.1 A function $f : \mathbb{X} \to \mathbb{R}$ is called $K$-convex, where $K \geq 0$, if for each $x \leq y$ and for each $\lambda \in (0, 1)$,

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y) + AK.$$

Suppose $f$ is a lower semi-continuous $K$-convex function, such that $f(x) \to \infty$ as $|x| \to \infty$. Let

$$S \in \arg \min_{x \in \mathbb{X}} \{f(x)\},$$

(3.5)

$$s = \inf \{x \leq S : f(x) \leq K + f(S)\}.$$
where \( k_n \) is introduced in (2.1). Since \( 0 < k_n \leq +\infty \), then
\(-\infty \leq \alpha^* < 1.\)

Define the following function for all \( t \in \mathbb{N}_0 \) and \( \alpha \geq 0, \)
\[
f_{t, \alpha}(x) := cx + \sum_{i=0}^{t} \alpha^i E[h(x - S_{i+1})], \quad x \in \mathbb{X}. \tag{4.2}
\]

Let \( f_{t, \alpha}(-\infty) = \lim_{t \to -\infty} f_{t, \alpha}(x) \) and
\[
N_\alpha := \inf \{ t \in \mathbb{N}_0 : f_{t, \alpha}(-\infty) = +\infty \}, \tag{4.3}
\]
where the infimum of an empty set is \(+\infty\).

The following theorem provides the complete description of optimal finite-horizon policies for all discount factors \( \alpha. \)

**Theorem 4.1.** Let \( \alpha > 0. \) Consider \( \alpha^* \) defined in (4.1). If \( \alpha^* < 0 \) (that is, Condition 3.3 holds), then the statement of Theorem 3.4(i) holds. If \( 0 \leq \alpha^* < 1, \) then the following statements hold for the finite-horizon problem with the discount factor \( \alpha: \)

(i) if \( \alpha \in [0, \alpha^*], \) then a policy that never orders is optimal for every finite horizon \( N = 1, 2, \ldots; \)

(ii) if \( \alpha > \alpha^* \), then \( N_\alpha < +\infty \) and for a finite horizon \( N = 1, 2, \ldots; \) the following is true:

(a) if \( N \leq N_\alpha, \) then a policy that never orders at steps \( t = 0, 1, \ldots, N - 1 \) is optimal;

(b) if \( N > N_\alpha, \) then a policy that never orders at steps \( t = N - N_\alpha, \ldots, N - 1 \) and follow \( (s_{N-t-1, \alpha}, S_{N-t-1, \alpha}) \) policy at steps \( t = 0, \ldots, N - N_\alpha - 1 \) is optimal, where the real numbers \( S_{t, \alpha} \) satisfy (3.5) and \( s_{t, \alpha} \) are defined in (3.6) with \( f(x) := G_{t, \alpha}(x), \) \( x \in \mathbb{X}. \)

The following theorem provides the complete description of optimal infinite horizon policies for all discount factors \( \alpha. \)

**Theorem 4.2.** Let \( \alpha \in [0, 1). \) Consider \( \alpha^* \) defined in (4.1). The following statements hold for the infinite-horizon problem with the discount factor \( \alpha: \)

(i) if \( \alpha^* < \alpha, \) then an \( (s_\infty, S_\infty) \) policy is optimal, where the real numbers \( S_\infty \) and \( s_\infty \) are defined in (3.5) and (3.6) respectively with \( f(x) := G_{\alpha}(x), \) \( x \in \mathbb{X.} \) Furthermore, a sequence of pairs \( (s_{t, \alpha}, S_{t, \alpha})_{t=N_\alpha-N_\alpha+1} \) considered in Theorem 4.1 (iii, b) is bounded, and, for \( (s_\infty, S_\infty) \) is a limit point of the sequence, then the \( (s_\infty, S_\infty) \) policy is optimal for the infinite-horizon problem with the discount factor \( \alpha; \)

(ii) if \( \alpha^* \geq \alpha, \) then the policy that never orders is optimal.

5. **CONTINUITY OF THE VALUE FUNCTIONS**

In this section we show that the value functions \( v_{N, \alpha}(x), \) \( N = 1, 2, \ldots, \) and \( v_{\alpha}(x) \) are continuous in \( x \in \mathbb{X}. \) The following theorem describes continuity of value functions for finite-horizon problems considered in this paper.

**Theorem 5.1.** For \( N = 1, 2, \ldots, \) consider a \( N \)-horizon inventory control problem. The functions \( v_{N, \alpha}(x) \) and \( G_{t, \alpha}(x), t = 0, 1, \ldots, N, \) are continuous on \( \mathbb{X} \) for all \( \alpha \geq 0. \)

The following theorem describes continuity of value functions for infinite-horizon problems considered in this paper.

**Theorem 5.2.** Consider an infinite-horizon inventory control problem with expected total discounted cost criterion. The functions \( v_{\alpha}(x) \) and \( G_{\alpha}(x) \) are continuous on \( \mathbb{X} \) for all \( \alpha \in [0, 1). \)

Theorems 5.1 and 5.2 imply the following corollary.

**Corollary 5.3.** The statements of Theorems 3.4, 4.1, and 4.2 remain correct, if the second sentence of Definition 3.2 is modified in the following way: a policy is called an \( (s_1, S_1) \) policy at step \( t, \) if it orders up to the level \( S_1, \) if \( x_t < s_1, \) does not order, if \( x_t > s_1, \) and either does not order or orders up to the level \( S_1, \) if \( x_t = s_1. \)

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6. **REFERENCES**


