Agreement in Spins and Social Networks

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Abstract We introduce a model of mutually influencing individuals who hold one of two opinions about some matter. This is similar to a system of particles that can have one of two spin states. The model considers N sub-systems where individuals stay for some time, and then move from one sub-network to another independently of each other, or they may leave the overall network from any one of the sub-systems. They arrive externally to any sub-system according to independent Poisson processes. In each sub-system individuals can influence each other to align with their own opinion or spin, and their opinion can also fluctuate at random in either opinion. The system also allows for a bias or directional field in any of the sub-systems that influences the individuals or spins that are locally present. We show that even with a weak bias, when random fluctuations become small then all the individuals or spins in a given sub-network will align with probability one, to the opinion or spin direction represented by the bias or spin.

1. INTRODUCTION

It has been long known that similarity breeds connection [4], known as homophily or assortativity [5, 6]. Here we deal with the complementary issue of similarity that is *breeded by* connection. We consider binary opinions, and study the effect of mutual influence among individuals that share the same neighborhood or site for some time. Mutual persuasion through a mutually influencing random field and the effect of a"bias" field in the site where individuals encounter each other, as well random symmetric switches of opinion are considered. In particular we show that a small bias field suffices to convince individuals to "jell' around the same opinion when either the noise is small compared to other effects such as bias and mutual persuasion, or when the number of individuals becomes large and noise disappears faster than the other effects in the system.

Consider a network of N "sites" in which particles which have a positive or negative spin congregate for some time. Each particle can move among these sites independently of other particles. For simplicity in the sequel we will say "the particle's polarity" as well as "the particle's spin polarity", to mean the same thing. In some (but not all) sites a particle's polarity is influenced by three factors: (a) A local field which favours one of the spin directions, inducing opposite spin particles to switch polarity. (b) A noise effect which induces spins to switch polarity in either direction, and (c) The attraction from particles at the site that have the opposite spin which can induce a switch of a particle's polarity. (d) We also allow the particles to switch polarity probabilistically as they move from one site to another.

We will assume that for any of the sites $i,\ j \in \{1,\ \dots,N\}$ the positive particles move from site i to site j with probability p^{++}_{ij} without changing polarity, while with probability p^{+-}_{ij} they move from i to j and switch polarity from positive to negative. Similarly we have the probabilities $p^{--}_{ij},\ p^{-+}_{ij}$ for changing sites and switching polarities for the negative polarity particles, and from any site i they may also leave the system with probabilities $d^+_i,\ d^-_i$ so that $\sum_{j=1}^{N}[p^{x+}_{ij}+p^{x-}+d^x_i=1$ for $x\in\{+,-\}$. Furthermore each particle remains in site i, independently of other particles in that site, for an exponentially distributed time of average value $[\mu_i]^{-1}$. Positive and negative polarity particles arrive to site i either from some other sites, as already indicated, or they arrive from an outside source according to independent Poisson processes of parameters (rates) $\lambda^+_i,\ \lambda^-_i$, respectively.

Let $V_i(t),\ v_i(t)$ denote the number of positive and negative polarity particles, respectively, at site i at time $t\geq 0$. Denote the vectors of the total number of particles $k = (k_1,\ \dots,k_N)$ where $k_i = v_i + V_i$.

Also denote by $o(\Delta t)$ some function that decreases to zero faster than Δt, i.e. that satisfies: $\lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}=0$. Then in the time interval $[t, t+\Delta t[$: $[\phi_{i,k_i,V_i}\Delta t]v_i(t)+o(\Delta t)$ is the probability that a negative polarity particle will switch to positive polarity, where $\phi_{i,k_i,V_i}\geq 0$ is the positive field strength at site i when the number of positive polarity particles is $V_i(t)=V_i$ and the total number of particles at the site is $k_i{=}v_i+V_i$. We see that the field strength ϕ_{i,k_i,V_i} acts on each individual negative polarity particle, and that its strength depends on V_i and k_i. Thus we can even have situations where the field is switched off as soon as the number of **positive** polarity particles rises **above** zero, i.e. $\phi_{i,k_i,0}>0$, and $\phi_{i,k_i,V_i}=0$ when $V_i>0$.

$[\tau_{i,k_i}\Delta t]v_i(t)+o(\Delta t)$ is the probability that a negative polarity particle switches to positive polarity, and with probability $\tau_{i,k_i}\Delta t V_i(t)+o(\Delta t)$ switches to negative polarity, due to thermal noise of intensity τ_{i,k_i}, where $k_i=v_i+V_i$ is the total number of particles at site i. Note that in this notation $\tau_{i,k_i}\to 0$ stands for a model that is being *cooled*, while $\tau_{i,k_i}\to\infty$ is *very high temperature*, and we can think of τ_{i,k_i} as a parameter that is proportional to the "temperature" of site i .

$[\alpha_{i,k_i}\Delta t]v_i(t)V_i(t)+o(\Delta t)$ is the probability that at site i a negative polarity particle switches to positive polarity; note that the switching probability is proportional to the number of negative polarity particles $v_i(t)$ (since some of them may switch) and to the number of positive polarity particles that try to influence the others. Similarly, a positive polarity particle switches to negative polarity with probability $\alpha_{i,k_i}\Delta t V_i(t) v_i(t) + o(\Delta t)$.

Let $\nu(t) = (v_1(t), V_1(t), \ldots, v_N(t), V_N(t))$ denote the state vector for the system at time $t \geq 0$. By a slight abuse of notation, also choose $\nu = (v_1, V_1, \ldots, v_N, V_N)$ to be a particular value taken by the random process $\nu(t)$. We define the following notation:

$$\nu^+_{i-} = (v_1, V_1, \ .. \ , v_i, V_i - 1, \ .. \ , v_N, V_N),$$
$$\nu^-_{i-} = (v_1, V_1, \ .. \ , v_i - 1, V_i, \ .. \ , v_N, V_N),$$
$$\nu^+_{i,+} = (v_1, V_1, \ .. \ , v_i, V_i + 1, \ .. \ , v_N, V_N),$$
$$\nu^-_{i,+} = (v_1, V_1, \ .. \ , v_i + 1, V_i, \ .. \ , v_N, V_N),$$
$$\nu^{++}_{j+,i-} = (v_1, V_1, \ .. \ , v_i, V_i - 1, \ .. \ , v_j, V_j + 1, \ .. \ , v_N, V_N),$$
$$\nu^{--}_{j+,i-} = (v_1, V_1, \ .. \ , v_i - 1, V_i, \ .. \ , v_j + 1, V_j, \ .. \ , v_N, V_N),$$
$$\nu^{+-}_{i-,i+} = (v_1, V_1, \ .. \ , v_i + 1, V_i - 1, \ .. \ , v_N, V_N),$$
$$\nu^{+-}_{i+,i-} = v_1, V_1, \ .. \ , v_i - 1, V_i + 1, \ .. \ , v_N, V_N).$$

We also represent the joint probability of system state as:

$$p(\nu, t) \equiv Prob[\nu(t) = \nu]. \tag{1}$$

Based on the previous discussion, the Chapman-Kolmogorov [1] or Master Equations for the system can be written as follows, where to simplify the notation we have dropped the dependency of the various rates on k_i:

$$\begin{aligned}
\frac{d}{dt}p(\nu,t) = \sum_{i=1}^{N} &\left[(\lambda_i^+ p(\nu^+_{i-},t) + \lambda_i^- p(\nu^-_{i-},t))1[V_i > 0]\right. \\
&+\mu_i((V_i+1)p(\nu^+_{i,+},t)d_i^+ + (v_i+1)p(\nu^-_{i,+},t))d_i^- + \\
\sum_{j=1}^{N} &\mu_j[p_{ji}^{++}(V_j+1)p(\nu^{++}_{j+,i-},t) + p_{ji}^{--}(v_j+1)p(\nu^{--}_{j+,i-},t) \\
&+p_{ji}^{+-}(V_j+1)p(\nu^{+-}_{j+,i-},t) + p_{ji}^{-+}(v_j+1)p(\nu^{-+}_{j+,i-},t)] \\
&+\alpha_i(V_i-1)(v_i+1)p(\nu^{+-}_{i-,i+},t)1[V_i>0] \\
&+\alpha_i(v_i-1)(V_i+1)p(\nu^{+-}_{i+,i-},t))1[v_i>0] \\
&+\beta_i((v_i+1)p(\nu^{+-}_{i-,i+},t) + (V_i+1)p(\nu^{+-}_{i+,i-},t))] \\
&+\phi_{i,V_i-1}(v_i+1)p(\nu^{+-}_{i-,i+},t)1[V_i>0] \\
&-p(\nu,t)\left[\tau_{ik_i}(v_i+V_i) + \phi_{i,V_i}v_i + 2\alpha_i v_i V_i\right. \\
&\left.+\mu_i(v_i+V_i) + \lambda_i^+ + \lambda_i^-\right].
\end{aligned} \tag{2}$$

Now define:

$$k_{i+} = k = (k_1, \ \ldots \ , k_i + 1, \ \ldots \ , k_N),$$
$$k_{i-} = (k_1, \ \ldots \ , k_i - 1, \ \ldots \ , k_N),$$
$$k_{i+,j-} = (k_1, \ \ldots \ , k_i + 1, \ \ldots \ , k_j - 1, \ \ldots \ , k_N),$$

and write $\lambda_i = \lambda_i^+ + \lambda_i^-$ for the total arrival rates of particles to each site from sources external to the network. The equations (2) can be reduced to the case where only changes in the state $k(t)$, representing the total number of particles in each site, are considered and they satisfy Jackson's theorem [2] so that when the probability matrix $\{p_i j\}_{[N\times N]}$ is transient, then the flow equations:

$$\Lambda_i = \lambda_i + \sum_{j=1}^{N} \Lambda_j p_{ji}, \ i = 1, \ \ldots \ , N \tag{3}$$

yield the total number of particles arriving to each of the N sites per unit time, and the joint probability distribution of the total number of particles in each site is the product of the marginal probability distributions at each site: $p(k) \equiv lim_{t\to\infty} p(k,t) = \Pi_{i=1}^N \frac{(\rho_i)^{k_i}}{k_i!} e^{-\rho_i}$, where $\rho_i = \frac{\Lambda_i}{\mu_i}$.

2. SLOW EXTERNAL ARRIVALS AND DEPARTURES

Now consider the system with restrictions on the speed at which state changes occur at each site $1 \leq i \leq N$: State transitions that change the polarity of particles inside each site are much faster than transitions that change the total number of particles in each individual site, i.e. (a) $\lambda_i^+, \ \lambda_i^-, \ \mu_i(1-p_{ii}) << \alpha_i, \ \phi_{i,0}, \ \tau_{ik_i}$ In addition, we suppose that $\alpha_i >> \phi_{i,0}, \ \tau_{ik_i}$, and (b) For $V_i > 0$ we have $\phi_{i,V_i} = 0$. The last assumption simply states that the bias field ϕ_{i,V_i} at each site is only turned on when there are no positive polarity particles.

Define the conditional probability of the number of positive particles at site i given that the total number of particles at that site is k:

$$P_{i,k}(V,t) = Prob[\ V_i(t) = V \mid k_i(t) = k\ ],$$

Based on the assumptions (a) and (b), for $k > 0$ we write the local master equation for the number of positive polarity particles at site i, conditioned on the fixed k total number of particles at the site, as:

$$\begin{aligned}
\frac{d}{dt}P_{i,k}(V,t) = &-P_{i,k}(V,t)[2\alpha_i V(k-V) \\
&+\phi_{i,V}(k-V) + \tau_{ik_i}k] \\
+P_{i,k}(V+1,t)&1[V \leq k-1][\alpha_i(V+1)(k-V-1) \\
&+\tau_{ik_i}(V+1)] \\
+P_{i,k}(V-1,t)&1[V>0][\alpha_i(V-1)(k-V+1) \\
&+\tau_{ik_i}(k-V+1) + \phi_{i,V-1}(k-V+1)]
\end{aligned}$$

and its stationary solution $P_{i,k}(V)$ for fixed k obtained by setting $\frac{d}{dt}P_{i,k}(V,t) = 0$ is:

$$P_{i,k}(V) = P_{i,k}(0)\Pi_{l=0}^{V-1}\frac{\lambda_i(l)}{\mu_i(l+1)}, \ where$$
$$for\ k \geq 1: \ \lambda_i(0) = (\tau_{ik_i} + \phi_{i,0})k, \ \mu_i(k) = \tau_{ik_i}k,$$
$$\lambda_i(l) = \alpha_{i,k}l(k-l)[1 + \frac{\tau_{ik_i} + \phi_{i,l}}{\alpha_{i,k}l}], \ 1 \leq l \leq k-1,$$
$$\mu_i(l) = \alpha_{i,k}l(k-l)[1 + \frac{\tau_{ik_i}}{\alpha_i(k-l)}], \ 1 \leq l \leq k-1.$$

Therefore the probability that *all* the particles at site i have positive spin is:

$$P_{i,k}(k) = P_{i,k}(0)[1 + \frac{\phi_{i,0}}{\tau_{ik_i}}]\Pi_{l=1}^{k-1}[1 + \frac{\phi_{i,l}}{\alpha_{i,k}l + \tau_{ik_i}}], \tag{4}$$

and for for $1 \leq V \leq k-1$ we have:

$$P_{i,k}(V) = P_{i,k}(0)\frac{k(\tau_{i,k} + \phi_{i,k,0})}{\alpha_{i,k}V(k-V) + \tau_{i,k}V} \cdot \Pi_{l=1}^{V-1}\frac{\alpha_{i,k} + \frac{\tau_{i,k}+\phi_{i,k,l}}{l}}{\alpha_{i,k} + \frac{\tau_{i,k}}{k-l}}, \tag{5}$$

where the normalising constant, which is the probability that there are *no* positive spin particles at site i, is:

$$P_{i,k}(0) = [\ 1 + [1 + \frac{\phi_{i,0}}{\tau_{i,k}}]\Pi_{l=1}^{k-1}[1 + \frac{\phi_{i,l}}{\alpha_{i,k}l + \tau_{i,k}}] \quad (6)$$

$$+ \sum_{V=1}^{k-1} \frac{k(\tau_{i,k} + \phi_{i,k,0})}{\alpha_{i,k}V(k-V) + \tau_{i,k}V}\Pi_{l=1}^{V-1}\frac{\alpha_{i,k} + \frac{\tau_{i,k}+\phi_{i,k,l}}{l}}{\alpha_{i,k} + \frac{\tau_{i,k}}{k-l}}]^{-1}.$$

Now combining (4) and (6) we obtain:

$$P_{i,k}(k) = [\ \frac{\tau_{i,k}}{\tau_{ik_i} + \phi_{i,0}}\Pi_{l=1}^{k-1}\frac{\alpha_{i,k}l + \tau_{i,k}}{\alpha_{i,k}l + \tau_{i,k} + \phi_{i,k,l}} + 1 \quad (7)$$

$$+ \frac{\tau_{i,k}}{\alpha_{i,k}}\Pi_{l=1}^{k-1}\frac{\alpha_{i,k}l + \tau_{i,k}}{\alpha_{i,k}l + \tau_{i,k} + \phi_{i,k,l}}\sum_{V=1}^{k-1}[\frac{k}{V(k-V+\frac{\tau_{i,k}}{\alpha_{i,k}})}.$$

$$.\Pi_{l=1}^{V-1}\frac{\alpha_{i,k} + \frac{\tau_{i,k}+\phi_{i,k,l}}{l}}{\alpha_{i,k} + \frac{\tau_{i,k}}{k-l}}]]^{-1}.$$

3. WHEN THE BIAS ONLY ACTS WHEN ALL ARE AGAINST IT

Now assume that the bias only acts when **all** the spins or agents present an state or "opinion" which is in **opposition to the bias**. This is represented mathematically by the parameters $\pi_{i,k,l} = 0$ for $l > 0$, and $\phi_{i,k,0} > 0$.

Then from (7) we have:

$$P_{i,k}(k) = [\ \frac{\tau_{i,k}}{\tau_{ik_i} + \phi_{i,0}} + 1 \quad (8)$$

$$+ \frac{\tau_{i,k}}{\alpha_{i,k}}\sum_{V=1}^{k-1}\frac{k}{V(k-V+\frac{\tau_{i,k}}{\alpha_{i,k}})}\Pi_{l=1}^{V-1}\frac{1 + \frac{\tau_{i,k}+\phi_{i,k,l}}{\alpha_{i,k}l}}{1 + \frac{\tau_{i,k}}{\alpha_{i,k}(k-l)}}]^{-1}.$$

Furthermore, assume that $\tau_{i,k} << \phi_{i,k,0}$, $\tau_{i,k} << \alpha_{i,k}$.

The first order approximation of (8) correct up to order $\frac{\tau_{i,k}}{\alpha_{i,k}}$ and $\frac{\tau_{i,k}}{\phi_{i,k,0}}$ is:

$$P_{i,k}(k) \approx 1 - \frac{\tau_{i,k}}{\phi_{i,0}} \quad (9)$$

$$- \frac{\tau_{i,k}}{\alpha_{i,k}}\sum_{V=1}^{k-1}\frac{k}{V(k-V)}(1 - \frac{\tau_{i,k}}{V(k-V)\alpha_{i,k}}).$$

$$.(1 + \frac{\tau_{i,k} + \phi_{i,k,l}}{\alpha_{i,k}l})(1 - \frac{\tau_{i,k}}{\alpha_{i,k}(k-l)}),$$

hence:

$$P_{i,k}(k) \approx 1 - \frac{\tau_{i,k}}{\phi_{i,0}} \quad (10)$$

$$- \frac{\tau_{i,k}}{\alpha_{i,k}}\sum_{V=1}^{k-1}\frac{k}{V(k-V)},$$

$$\approx 1 - \frac{\tau_{i,k}}{\phi_{i,0}} - \frac{\tau_{i,k}}{\alpha_{i,k}}\sum_{V=1}^{k-1}[\frac{1}{V} + \frac{1}{k-V}],$$

$$\approx 1 - \frac{\tau_{i,k}}{\phi_{i,0}} - \frac{2\tau_{i,k}}{\alpha_{i,k}}\ln k - 1.$$

As a consequence:

Result 1 If $\alpha_{i,k} > 0$. $\phi_{i,k,0} > 0$, then:

$$\lim_{t\to\infty} P_{i,k}(t,k) = 1. \quad (11)$$

Result 2 If $\phi_{i,k,0} > 0$, and

$$\lim_{k\to\infty}\frac{\tau_{i,k}\ln(k-1)}{\alpha_{i,k}} = 0, \quad (12)$$

then:

$$\lim_{k\to\infty}\lim_{t\to\infty} P_{i,k}(t,k) = 1. \quad (13)$$

Result 3 If $\alpha_{i,k} > 0$, and

$$\lim_{k\to\infty}\frac{\tau_{i,k}}{\phi_{i,k,0}} = 0, \quad (14)$$

then:

$$\lim_{t\to\infty} P_{i,k}(t,k) = 1. \quad (15)$$

4. CONCLUSIONS

The work in this note was actually motivated by our work on very low power communications using spin effects of particles [3, 8]. The "agreement" effect discussed here can be a source of "noise" when bias fields are introduced in such a system. However bias can also be used to encode the binary data, in which case we will be interested in the resulting error for each individual particle. The overall model considered is then a network in which each "sub-network" is a device where particles aggregate and are encoded before they are transmitted to another node that must decode them to determine their binary content. Further work in this direction will combine our earlier paper [8] and the results that we present here.

Another interesting question is the role of the mutual field between particles, or between individuals in a social network, and how the size of each sub-network and its probability distribution, will affect the outcome.

5. REFERENCES

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