On Network Criticality in Wireless Networks

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ABSTRACT
Network criticality is a graph-theoretic metric that quantifies network robustness, and that was originally designed to capture the effect of environmental changes in core networks. This paper investigates the application of network criticality in designing robust power allocation and flow assignment algorithms for wireless networks. Achieving robust behavior in wireless networks is a challenging task due to constant changes in channel conditions and the interference. We consider network criticality as a natural robustness metric, and propose approaches to preserve the useful convexity properties of network criticality, while resolving issues related to the non-convexity of Shannon's capacity.

1. INTRODUCTION
Due to the time-varying nature of the wireless channels, the design of resource allocation methods in wireless networks significantly differs from that in wireline networks. Channel variations make it necessary to design robust algorithms for power allocation and flow assignment. The robust design of wireless networks has been tackled from different standpoints. For example in [1], the rate maximization in cognitive radio networks (CRN) is considered. It is shown that direct maximization of user rates results in large ripples and instability in the achievable rates of individual users. The authors in [1] proposed a max-min method to design a robust rate maximization method. The proposed algorithm in [1] removes instability; however, the total achievable rate is significantly reduced in comparison to the original case.

Network criticality is a robustness metric to capture the effect of environmental changes such as traffic variation and topology changes in networks. A network is modeled as a weighted graph, where the weights of a link denote the capacity or desirability of a path. Network criticality is defined as the average random-walk betweenness of a link (node) normalized by its weight. This quantity is independent of link (node) location and it is a decreasing and strictly convex function of link weights. Network criticality can be written in terms of the components of the undirected Moore-Penrose Laplacian matrix [2]:

$$\hat{\tau} = \frac{2}{n-1} \text{Tr}(L^+)$$

There is a useful interpretation of network criticality in terms of electrical circuits: network criticality is the unweighted average of the equivalent resistances in a network of resistors. Therefore optimizing criticality is equivalent to minimizing the average resistance or maximizing the average conductance of a network, which explains why network criticality can be considered as a global robustness metric. Furthermore, according to Thomson's principle from physics, Kirchhoff's equations yield the set of currents that minimize the overall power consumed in a resistive network [3]. Clearly, algorithms based on network criticality attempt to route network flows similarly to the way electric currents flow through an electrical network.

Network criticality was designed for wireline core networks, but it is conceptually applicable wherever the network is represented with a weighted graph in which the link weights are well defined. In this paper, we investigate appropriate definitions of link weights that allow us to use network criticality as an objective while preserving its convexity. In the following discussion, we assume that we have a wireless network in which the capacity of a link depends on not only the power allocated to itself, but also the powers allocated to the other links due to interference.

Throughout, we assume that the topology of a wireless network is given by a directed graph $G(N,E,W)$ (in most general form, it is a full-mesh, since potentially all the wireless nodes can send/receive traffic to/from all other nodes), where $N$, $E$, and $W$ denote the link set, node set, and link weight matrix respectively. The weight matrix is in general asymmetric; however, in calculating the network criticality we use an undirected symmetric matrix of the graph defined as $W_{sym} = \frac{W + W^T}{2}$, where $W^T$ denotes the transpose of $W$. Clearly $W_{sym}$ is a symmetric matrix.

2. SERVICE TIME CONTROL
Let $G = [g_{ij}]$ be the channel gain matrix of the wireless network, where $g_{ij}$ is the power gain from the transmitter of link $i$ to the receiver of link $j$. In an ideal interference-free environment only the diagonal entries of $G$ (i.e. $g_{ii}$ for $i \in E$, where $E$ is the set of wireless links) are nonzero, but in a typical wireless network the off-diagonal terms $g_{ij}$ for $i \neq j$
are nonzero and due to interference. In order to model the variability of channel gains, we adopt a probabilistic framework in which \( G \in \{ G_1, G_2, ..., G_k \} \), where each channel gain matrix \( G_u \) (\( u \) is referred to as the channel gain state) exists with probability \( p_u \). In such environments and if there is no prior interference cancellation mechanism in place, the upper bound for achievable rate is given by Shannon’s formula:

\[
    r_{i,u} \leq \log(1 + \frac{g_{i,u}p_{i,u}}{\sigma_i + \sum_{k \neq i} g_{k,i}p_{k,u}}) \tag{1}
\]

where \( r_{i,u} \) and \( p_{i,u} \) denote the rate (bits per channel use) and power of the transmitter of wireless link \( i \) at channel gain state \( u \) respectively. Furthermore, let \( p_{i,u} \) denote the transmit power of user \( i \) in channel gain state \( u \). We assume the transmitter of link \( i \) has the transmit power constraint of \( P_i^{max} \), i.e. \( 0 \leq p_{i,u} \leq P_i^{max} \). Suppose at a known channel gain state \( u \), the transmitter of link \( i \) wishes to send equal-length packets of length \( \gamma_i \) to the receiver of link \( i \). Let \( t_{i,u} \) be the number of channel uses or the service time of each packet, then:

\[
    t_{i,u} \geq \frac{\gamma_i}{r_{i,u}} \tag{2}
\]

We introduce variable \( x_{i,u} \) as the maximum SINR (signal to interference plus noise ratio) at the receiver of link \( i \). Hence:

\[
    x_{i,u} \leq \frac{g_{i,u}p_{i,u}}{\sigma_i + \sum_{k \neq i} g_{k,i}p_{k,u}} \tag{3}
\]

Combining inequalities (2) and (3), we have:

\[
    \frac{\gamma_i}{\log(1 + x_{i,u})} \leq t_{i,u} \tag{4}
\]

Our goal is to find to construct a convex optimization problem in order to find the optimal values for service times and user powers to minimize the network criticality (considering (2) and (4) as constraints of the optimization problem); however, in their present form inequalities (2) and (4) do not impose convex sets. To find an equivalent convex form, we rewrite constraint (3) as:

\[
    \frac{x_{i,u} \sigma_i}{g_{i,u}p_{i,u}} + \sum_{k \neq i} \frac{x_{i,u}g_{k,i}p_{k,u}}{g_{i,u}p_{i,u}} \leq 1 \tag{5}
\]

We apply the following change of variables:

\[
    y_{i,u} = \ln(x_{i,u}), \quad q_{i,u} = \ln(p_{i,u}) \tag{6}
\]

Equation \( q_i = \ln(p_{i,u}) \) involves a slight loss of generality, because we should have \( p_{i,u} > 0 \), but we will see that it is worth since it provides convexity. Applying equation (6) in (4) and (5) results in:

\[
    \frac{\gamma_i}{\log(1 + e^{y_{i,u}})} \leq t_{i,u} \tag{7}
\]

\[
    \ln\{ e^{y_{i,u} - q_{i,u} + \ln(q_{i,u}) - \ln(q_{i,u} + \ln(q_{i,u}))} + \sum_{k \neq i} e^{y_{k,u} - q_{k,u} + \ln(q_{k,u}) - \ln(q_{k,u} + \ln(q_{k,u}))} \} \leq 0 \tag{8}
\]

Convexity of constraint (7) can be examined by evaluating the second derivative of function \( f(x) = \frac{\ln(2)}{\ln(1 + e^x)} \) with respect to \( x \). Note that \( f(x) = \frac{\ln(2)}{\ln(1 + e^x)} \), therefore, we have:

\[
    \frac{d^2}{dx^2} f(x) = -\ln(2) \frac{e^x}{(1 + e^x)^2} (1 - \frac{2e^x}{\ln(1 + e^x)}) f^2(x) \tag{9}
\]

Equation (9) shows that \( f(x) \) is a decreasing function of \( x \). By finding the derivative of both sides in equation (9) and after some simplification, we have:

\[
    \frac{d^2}{dx^2} f(x) = -\ln(2) \frac{e^x}{(1 + e^x)^2} \left(1 - \frac{2e^x}{\ln(1 + e^x)}\right) f^2(x) \tag{10}
\]

Considering the fact that \( e^x > 0 \), we have \( e^x > \ln(1 + e^x) \), hence we conclude that:

\[
    1 - \frac{2e^x}{\ln(1 + e^x)} < -1
\]

\[
    \frac{d^2}{dx^2} f(x) > \ln(2) \frac{e^x}{(1 + e^x)^2} f^2(x) > 0 \tag{11}
\]

The second derivative of \( f(x) \) according to (11) is nonnegative; therefore, constraint (7) represents a convex set. Furthermore, the left hand side of constraint (8) is of the form Log-Exp-Sum which is also convex \([4]\).

We are now ready to formulate our robust convex optimization problem using network criticality. There are different approaches to construct such an optimization problem. The general idea is to define weights for the graph such that an increase in the weight increases the desirability of the link, while at the same time preserving the convexity of network criticality with respect to the variables. We will discuss some examples of such weight assignments for different purposes.

First, we define link weights as a concave decreasing function of the mean service time over all possible channel gain states: \( \bar{t}_i = \sum \bar{p}_u t_{i,u} \)'s (service times), i.e. \( W_{sym} = \Phi(\bar{t}_i) \), where \( \Phi \) is a concave decreasing function of \( \bar{t}_i \). We symmetrize the weight matrix (denoted by \( W_{sym} \)) as discussed in section 1, and we use it to calculate \( \bar{t} \). Then, the robust convex optimization will be:

\[
\begin{align*}
    \text{minimize} & \quad \hat{\tau}(W_{sym}) \\
    \text{subject to} & \quad (\forall i \in E, \forall u \in \{1, 2, ..., k\} : ) \\
    & \quad \frac{\gamma_i}{\log(1 + e^{y_{i,u}})} \leq t_{i,u} \\
    & \quad \ln\{ e^{y_{i,u} - q_{i,u} + \ln(q_{i,u}) - \ln(q_{i,u} + \ln(q_{i,u}))} + \sum_{k \neq i} e^{y_{k,u} - q_{k,u} + \ln(q_{k,u}) - \ln(q_{k,u} + \ln(q_{k,u}))} \} \leq 0 \\
    & \quad q_{i,u} \leq \ln(P_i) \tag{12}
\end{align*}
\]

Note that since \( \Phi \) is a concave function of \( \bar{t}_i \)'s and \( \hat{\tau} \) is a convex decreasing function of weights \([2]\), network criticality \( (\bar{t}) \) is a convex function of \( \bar{t}_i \)'s \([4]\). Moreover, optimization problem (12) is valid for the whole range of SINR values (low SINR regime up to high SINR regime).

We discuss a second robust optimization problem, where our goal is to provide robustness with regards to the SINR. To this end, we define the weight of a link \( i \) for the channel gain state \( u \) as \( w_{i,u} = y_{i,u} \) (note that increasing \( y_{i,u} \) will improve desirability of link \( i \)), and after symmetrizing the weight matrix, we let the mean of network criticality over all channel gain states be the objective function of the optimization problem. Let \( \hat{\tau} \) denote the network criticality for channel gain state \( u \), then the optimization problem can be
summarized as follows:

\[
\begin{align*}
\text{minimize} & \quad \sum_u \hat{p}_u \tau_u \\
\text{subject to} & \quad (\forall i \in E, \forall u \in \{1, 2, ..., k\} :)
\end{align*}
\]

\[
\begin{align*}
& \log(1 + e^{\gamma_{i,u}^c}) \leq l_{i,u} \\
& \ln\{e^{q_{i,u} - q_{i,u} - \ln(g_{i,u})} + \ln(c_{i,u}) + \sum_{k \neq i} e^{q_{k,u} - q_{i,u} + q_{k,u} - \ln(g_{k,u})} + \ln(g_{k,u})\} \leq 0 \\
& q_{i,u} \leq \ln(p_{i,u}^{max}) \\
& q_{i,u} \geq 0
\end{align*}
\]

The last constraint (nonnegativity of \( q_{i,u} \)) is added to guarantee that the link weights are nonnegative (in order for network criticality to exist). In terms of SINR, the nonnegativity of \( q_{i,u} \) according to (4) means that signal-to-noise-interference ratio should be more than 1. This is true in most practical situations as usually \( g_{i,u} >> q_{i,u} \). Please note that we have not assumed that the system is in high SINR regime. The formulation is valid for low SINR regime as long as \( \text{SINR} > 1 \).

We can define link weights differently to achieve other objectives. Suppose, we need to have a robust power allocation scheme as the primary goal. Then we define the weight of link \( i \) as \( w_{i,u} = q_{i,u} \), and after symmetrizing the weight matrix, we calculate the mean of network criticality in optimization problem (13) based on that. The rest is similar to the optimization problem (13).

3. HIGH-SINR REGIME

The special case of high-SINR regime is easier and it is already considered in the literature. Let \( \gamma_{i} \) be the traffic flow matrix that needs to be routed in the wireless network. In [5], the joint robust resource allocation and power assignment problem is discussed. We adopt the problem introduced in [5] and provide a robust version of it. We would like to have robustness in the traffic distribution; therefore, we choose the available capacity of a communication link (i.e. the Shannon capacity of a link minus the traffic flow passing through the link) as the weight of our link: \( w_{i,u} = c_{i,u} - f_{i,u} \), where \( f_{i,u} \) is the total flow of link \( i \) for channel gain state \( u \), and \( c_{i,u} = \log(1 + \frac{g_{i,u}}{\sum_{k \neq i} g_{k,u} + c_{k,u}}) \).

Applying the change of variable \( q_{i,u} = \ln(p_{i,u}) \), we can approximate the wireless link capacity as follows [5]:

\[
c_{i,u} = -\log\left(\frac{g_{i,u}}{q_{i,u}} e^{-q_{i,u}} + \sum_{k \neq i} \frac{g_{k,u}}{q_{k,u}} e^{-q_{k,u} - q_{i,u}}\right)
\]

Equation (14) is a Log-Sum-Exp expression; therefore, it is a concave function (due to the negative sign) [4]. In order to write the flow conservation for the network, we define total incoming traffic to node \( d \) as \( \gamma^{(d)} = \sum_{i} \gamma_{i,d} \). Moreover, we define the flow of link \( i \) for destination \( d \) as \( f^{(d)}_{i} \). Then the flow conservation equations can be written in matrix form as \( Bf^{(d)} = \gamma^{(d)} \), where \( f^{(d)} \), \( \gamma^{(d)} \), and \( B \) denote the vector of link flows for node \( d \), the external input traffic vector for \( d \), and the graph incidence matrix respectively. Now suppose the goal is to minimize the total power assigned to all the wireless users, while we would like to robustly distribute the traffic flows. We can provide robustness by introducing a constraint to guarantee that the value of network criticality does not exceed a given threshold. The optimization problem for joint resource and power assignment is:

\[
\begin{align*}
\text{Minimize} & \quad \sum_u \hat{p}_u \tau_u \\
\text{Subject to} & \quad (\forall i \in E, \forall d \in N, \forall u \in \{1, 2, ..., k\} :)
\end{align*}
\]

\[
\begin{align*}
& \gamma^{(d)} = \sum_u \hat{p}_u q_{i,u} \\
& Bf^{(d)} = \gamma^{(d)} \\
& f_{i,u} \leq -\log\left(\frac{g_{i,u}}{q_{i,u}} e^{-q_{i,u}} + \sum_{k \neq i} \frac{g_{k,u}}{q_{k,u}} e^{-q_{k,u} - q_{i,u}}\right) - w_{i,q} \\
& f_{i,u} \geq 0 \\
& e^{q_{i,u}} \leq p_{i,u}^{max} \\
& \sum_u \hat{p}_u \tau_u \leq a
\end{align*}
\]

where \( a \) is a known upper bound for network criticality. Optimization problem (15) assigns user powers and distributes traffic flows such that the sum of power consumption is minimized while we make sure that network criticality remains below the specified upper bound (i.e. \( a \)), which guarantees the robustness in flow distribution. Lower constraint bound for network criticality makes the problem more robust but we expect to see more power required.

4. CONCLUSIONS AND ROAD MAP

In this paper we investigated a number of approaches to construct convex optimization problems to provide robust joint resource allocation and power assignment for wireless interference aware networks. We have shown how network criticality can be introduced to provide robust designs while maintaining desirable convexity properties.

We are in the initial steps of developing robust solutions for wireless networks using network criticality. In this paper we explained some theoretical methods that we have developed to construct convex optimization problems for joint power allocation and flow assignment. The next step is to test the proposed frameworks on real wireless systems. Our goal is to design robust algorithms based on the proposed optimization problems to do the power allocation and flow assignment in a distributed manner.

5. REFERENCES