An Approximate Product-form Solution and Bound for Loss Networks

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ABSTRACT

We present numerical results for an application of the Reversed Compound Agent Theorem (RCAT, [1]) to loss networks. RCAT can check for the existence of product-form solutions and provide an approximate model when a product-form solution cannot be found thus. Up until now, there are no existing product-form solutions for loss networks, according to the best knowledge of the author. The stationary distribution of a queueing network is generally computed as the normalized solution of a system of linear equations, called the global balance equations. With RCAT, we can obtain an approximate product-form solution easily by introducing invisible actions in some nodes that synchronize with, and so enable, certain transitions in other nodes. We show that our approximation provides an upper bound and our numerical results compare quantitatively this bound with the exact, direct Markov chain solution. The results show that the accuracy of the new model is good with moderate external arrival rates, i.e. that the bound is tight. This greatly facilitates efficient performance analysis and design-blueprints for computer or communication systems having loss network characteristics, for example, wireless sensor networks.

Categories and Subject Descriptors

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General Terms

Theory, Performance

Keywords

product-form solutions, approximate model, loss networks, reversed compound agent theorem

1. INTRODUCTION

Customer loss networks originated in networks with blocking [3] or finite capacity constraints [5, 4]. In this poster, we investigate models in which arriving customers are simply lost when a queue is full, i.e. has no remaining capacity. In general, performance analysis in queueing networks with customer losses is difficult as the steady state probability is not a product-form. Therefore, we attempt the approximation of loss networks through the RCAT methodology.

2. GENERAL LOSS NETWORKS

Consider an N-node, open loss network of $M/M/m_k/F_k$ queues with Poisson external arrivals at rate λ_k at each node k of finite capacity F_k, and a single class of customers ($1 \leq k \leq N$). Customers queue for service at node k which has m_k servers, each of which has constant service rate μ_k. The main feature of Loss Networks is that arrivals are lost, discarded from the network, when the queue is full, i.e. when the number of customers at node k is F_k. The state-dependent service rate in this loss model is

$$\mu_k(x) = \begin{cases} x\mu_k & x < m_k \\ m_k\mu_k & x \geq m_k \end{cases}$$

RCAT's conditions do not hold in this network since the passive actions (arrivals) are not enabled in the full state of a node, but by introducing invisible, passive actions in those states (to allow arrivals to be discarded) and correspondingly adding invisible, active actions with rate equal to that of the (common) reversed rate of the departures to other nodes, the stationary distribution is found to be

$$\pi(n_1, n_2, \cdots, n_N) \propto \begin{cases} \prod_{k=1}^{N} \frac{\rho_k^{n_k}}{n_k!} & n_k < m_k \\ \prod_{k=1}^{N} \frac{\rho_k^{n_k}}{m_k^{n_k - m_k} m_k!} & m_k \leq n_k \leq F_k \end{cases}$$

where n_k is the number of customers at node k, $\rho_k = \frac{\lambda_k + w_k}{\mu_k}$ and w_k is the sum of the internal arrival rates at node k, for $1 \leq k \leq N$.

The parameters of this product-form are actually unremarkable: it is precisely what we would obtain if we took the well known, Jackson, solution for queueing networks with unconstrained queue lengths and simply renormalised over the truncated state space that arises from the finite capacity queues. However, what is interesting is that this product-form is correct when losses are compensated for by extra arrivals at a node from each of its upstream nodes when and only when they are in their full states. Moreover, the net rate of the compensating departures from a node is equal to its loss rate.

3. UPPER BOUND OF APPROXIMATION

The difference between this approximate model and the original (exact) model is the introduction of invisible acitons. Consider a customer at some node j on a path through the network. When the previous node (i say) in the path is full, its invisible active action (in the approximate model only) generates extra arrivals in its successor nodes, with rate $p_{ij}\overline{\mu_i}$ at node j, where p_{ij} is the routing probability from node i to node j and $\overline{\mu_i}$ is the common *reversed* rate of the service rate at node i. In an N-node network, therefore,

$$\text{Total extra arrival rate at node } j = \sum_{i \neq j} P(n_i = F_i) \cdot \overline{\mu_i} p_{ij} = \sum_{i \neq j} P(n_i = F_i) \cdot x_{ij}$$

where $P(n_i = F_i)$ is the probability that queue i is full, F_i is the capacity of queue i, and x_{ij} is the internal traffic rate from node i to j – determined by assigning it to the corresponding passive action type in the application of RCAT and solving the rate equations [1].

This upper bound is unimpressive in itself, in that every node's total output is equal to its total input, so the corresponding lower bound on loss rate is zero. More importantly, the product-form shows that the queues in the bound-model behave independently, and we have a stochastic ordering on their stationary states. Thus, when we make arguments about throughput or loss rate bounds, we can use those (joint) probabilities (needed for the flux into a full state, for example), which are easy to compute from the product-form. In particular, the mean values shown in this poster (and higher moments too) will be upper bounds.

4. NUMERICAL RESULTS

We now compare the "exact" numerical results generated by direct solution of the Markov process underlying the original model of the actual loss network with the approximation based on RCAT described above [2]. Although we could use any loss network for this, to be concrete, we consider here a 3×3 grid with external arrivals to and departures from each node at its edges. The nodes in the grid are labelled sequentially by i, for $1 \leq i \leq 9$. Each node is connected in both directions to its adjacent nodes only.

In comparing the two models, we calculate the average percentage difference in the mean queue length across all the nodes. Table 1 shows that the upper bound and loss rate increase with increasing λ_i, and that the percentage difference between the two models is also getting bigger. This is not surprising since extra arrivals only ever originate at nodes that are full and this occurs more frequently as the network receives more load. Conversely, Tables 2 shows that the percentage difference is smaller when the numbers of servers m_i is increasing; again this is as expected since servers become less saturated with the increased service capacity. To conclude, based on these results and many others, some of which we will display on the poster, if accepted, we found that RCAT provides a good approximate product-form solution in grid loss networks with moderate external arrival rates. This clearly enhances the prospects for performance analysis in large networks of this kind since simulation (or exact solution of a Markov process) would become prohibitively expensive.

Table 1: Numerical comparison between the approximate and exact models: $m_i = 3$, $F_i = 4$, and $\mu_i = 0.9$

λ_i	Exact result	Loss rate	Upper bound	% difference
0.1	4.16e-1	3.929e-4	3.50e-3	0.4
0.2	8.07e-1	7.838e-3	7.07e-2	1.61
0.3	1.13	3.668e-2	3.41e-1	2.58
0.4	1.39	9.491e-2	9.27e-1	3.05

Table 2: Numerical comparison between the approximate and exact models: $\lambda_j = 0.1$, $F_i = 4$, and $\mu_i = 0.9$

m_i	Exact result	Loss rate	Upper bound	% difference
1	6.12e-1	6.051e-3	5.49e-2	7.11
2	4.29e-1	8.760e-4	7.82e-3	0.93
3	4.16e-1	3.928e-4	3.50e-3	0.40
4	4.15e-1	2.948e-4	2.63e-3	0.30

5. CONCLUSION

Customer loss networks are common in various kinds of computer and communication systems and their performance (and hence modelling thereof) is typically critical. However, simulation of large loss networks requires enormous computation time with a tremendous increase in the number of states arising from only a small increase in the dimensional parameters. In this poster, we introduced an approximate model with product-form solution through application of RCAT. With the product-form, the computation time of results is low even in very large networks and, moreover, the approximation is an upper bound on the exact solution. We are currently investigating lower bounds, but these are traditionally looser and harder to find. We currently do have another product-form approximation that does provide a lower bound except at low utilizations and is reasonably tight at high utilizations. Unfortunately, it is problematic to identify the threshold above which the result must be a lower bound.

6. REFERENCES

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