Heavy-traffic Analysis of the Generalized Switch under Multidimensional State Space Collapse

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ABSTRACT

The drift method was recently developed to study performance of queueing systems in heavy-traffic [1]. It has been used to analyze several queueing systems, including some where the Complete Resource Pooling (CRP) condition is not satisfied, like the input-queued switch [4]. In this paper we study the generalized switch operating under MaxWeight using the drift method. The generalized switch is a queueing system that was first introduced by [5], and can be thought of as extension of several single-hop queueing systems, such as the input-queued switch and ad hoc wireless networks. When the CRP condition is not satisfied, we prove that there is a multidimensional state space collapse to a cone and we compute bounds on a linear combination of the queue lengths that are tight in heavy-traffic. This work generalizes some of the results obtained by [1] and the results from [4], since the queueing systems studied there are particular cases of the generalized switch.

1. INTRODUCTION

In this paper we focus on the generalized switch, which was first introduced in [5] to model a wide class of single hop queueing systems in the literature. A generalized switch is a discrete time system with a finite number of queues, each with its own (independent) arrival process. Packets that arrive into each queue wait for service. The scheduling constraints in the system dictate that only a subset of queues can be served in each time slot. In each time slot, the scheduling problem is to pick the set of queues that are active. A well-known scheduling algorithm is MaxWeight, where the weight of each schedule is the sum of the lengths of the corresponding queues and the schedule with maximum weight is selected. Moreover, there are external factors that influence the set of feasible service rate vectors. We group all these factors in a single variable, that we call channel state. This channel state varies with time, and we model it as a random process which is independent of the arrival and the queue lengths processes.

The generalized switch can be considered as an extension of some general stochastic processing networks, such as the input queued switch, ad hoc wireless networks, parallel service systems, virtual machine scheduling for cloud computing etc. The input-queued switch is a discrete time model with n input ports and n output ports. Packets arrive to each input port with a predetermined output port where they should be processed. Each output port can process at most one packet in each time slot, and the processing rate is fixed and equal to one time slot. A usual model is to think of the switch as having n^2 queues, i.e. in each input port there is one separate queue for each output port. There are physical constraints that only allow the switch to send at most one packet from each input port and to process at most one packet in each output port, in each time slot. These constraints represent interference constraints as mentioned above for the generalized switch, and since they are fixed for all time slot, in the input-queued switch the channel state is fixed over time. The input-queued switch is one of the simplest single hop queueing systems that does not satisfy the CRP condition.

In this paper we focus on heavy-traffic analysis of the generalized switch. Heavy-traffic means that one takes a sequence of queueing systems where the arrival rates vector approaches the boundary of the capacity region. This regime has been extensively used in the literature, in the context of many different queueing systems. One of its advantages is that, in the limit, the system can be approximated by a queueing system in a lower-dimensional state space. This phenomenon is known as Stace Space Collapse (SSC). If SSC occurs into a one-dimensional subspace, then the system is said to satisfy the Complete Resource Pooling (CRP) condition. Heavy-traffic analysis of queueing systems was first developed by Kingman in the '60s, where he obtained the steady-state distribution of the scaled waiting time in heavy-traffic. The approach used by Kingman is based on using diffusion limits. The idea of this approach is to scale time and the queue lengths, to prove process level convergence to a Reflected Brownian Motion. The limiting distribution is then computed and interchange of limits must be shown to complete the proof. In the literature, this approach has been extensively used in a variety of queueing systems that satisfy the CRP condition, such as the load balancing system. More general systems such as the input queued switch, generalized switch, ad hoc wireless networks etc are studied only when the CRP condition is satisfied. Systems that do not satisfy the CRP condition are studied only when the resulting distribution has a product form, such as the bandwidth sharing network [3]. However the generalized switch does not exhibit such a product form stationary distribution in general.

More recently, the drift method has been first developed in [1], where the authors proposed a new notion of SSC that uses Lyapunov drift arguments. The idea of this new notion of SSC is to decompose the vector of queue lengths in two vectors, according to the heavy-traffic behavior of the queueing system. Then, SSC is proved by upper bounding the norm of one of these vectors, which represents the error of the heavy-traffic approximation. Once SSC is established, one computes asymptotically tight bounds on the moments of scaled linear combinations of the queue lengths. These bounds are computed by setting to zero the drift of a carefully chosen test function in steady-state. In order to obtain tight bounds in the heavy-traffic limit, it is essential that the test function captures the geometry of SSC.

In [1], the drift method is developed for queueing systems that satisfy the CRP condition, such as a load balancing systems, ad hoc wireless networks and the generalized switch under CRP condition. Further, in [2] the authors use a novel view of the drift method to compute the steady-state distribution in heavy-traffic of the vector of queue lengths in systems that satisfy the CRP condition. When the CRP condition is not satisfied, the drift method has also been successfully used to compute the heavy-traffic limit of the first moment of the sum of the queue lengths. In particular, [4] applied this approach to an input-queued switch operating under MaxWeight.

In this paper, we use the drift method for the generalized switch when SSC occurs into a multi-dimensional subspace. The main contribution of this paper is to provide SSC proof and computation of asymptotically tight bounds on a linear combination of the queue lengths for one of the most general single hop queueing systems in the literature.

2. MODEL

Consider a generalized switch as described above. Arrivals to the i^{th} queue form a sequence $\{a_i(k) : \ k \geq 1\}$ of i.i.d. random variables, for each $i \in [n]$, where $[n] = \{1, \ldots, n\}$. For each $i \in [n]$, let $\lambda_i = \mathbb{E}[a_i(1)]$, $\sigma^2_{a_i} = \text{Var}[a_i(1)]$ and assume $a_i(1)$ is bounded with probability 1 for all $i \in [n]$. The arrival processes to different queues are independent. Let $s_i(k)$ be the offered service to queue i in time slot k, i.e. the number of packets from the i^{th} queue that would be processed if there are enough packets in line. Let $u_i(k)$ be the unused service in queue i in time slot k, i.e. the difference between the offered service and the number of packets that are actually processed in time slot k.

The interference constraints among servers depend on the channel state, that we model as a sequence of i.i.d. random variables $\{J(k) : \ k \geq 1\}$ where $J(k)$ represents the channel state in time slot k. If $J(k) = j$, all feasible service rate vectors are contained in the set $\mathcal{S}^{(j)}$. Assume that the random variables $J(k)$ have finite state space $\mathcal{J}$ and that for each $j \in \mathcal{J}$ the set $\mathcal{S}^{(j)}$ is finite. Therefore, the service rate vectors are bounded. Let $\boldsymbol{\psi}$ be the probability mass function of $J(1)$, i.e. for each $j \in \mathcal{J}$ we have $\psi_j = \mathbb{P}[J(1) = j]$.

In each time slot model, the order of events in one time slot is as follows. First, the channel state is observed, second a schedule is selected, third arrivals occur, and at the end of each time slot services occur. Therefore, the dynamics of the queues are as follows. For each $k \geq 1$ and $i \in [n]$

$$q_i(k+1) = q_i(k) + a_i(k) - s_i(k) + u_i(k). \tag{1}$$

In queue i, the unused service $u_i(k)$ is nonzero only when the respective service rate is greater than the number of packets in line (including arrivals). Therefore,

$$q_i(k+1)u_i(k) = 0 \qquad \forall k \geq 1 \quad \forall i \in \{1, \ldots, n\}. \tag{2}$$

However, if $i \neq j$, $q_i(k+1)u_j(k)$ is not necessarily zero.

In each time slot, the scheduling problem is solved using MaxWeight algorithm, i.e. the schedule with the longest total queue length is selected. Formally, if $J(k) = j$ then

$$\boldsymbol{s}(k) \in \arg \max_{\boldsymbol{x} \in \mathcal{S}^{(j)}} \langle \boldsymbol{q}(k), \boldsymbol{x} \rangle.$$

Ties are broken at random.

It is well known that the capacity region of the generalized switch is

$$\mathcal{C} = \sum_{j \in \mathcal{J}} \psi_j \, ConvexHull\left(\mathcal{S}^{(j)}\right) \tag{3}$$

and that MaxWeight is throughput optimal, i.e. the generalized switch operating under MaxWeight is positive recurrent for all $\boldsymbol{\lambda}$ in the interior of $\mathcal{C}$. Since the sets $\mathcal{J}$ and $\mathcal{S}^{(j)}$ $\forall j \in \mathcal{J}$ are finite, then $\mathcal{C}$ is a polytope in $\mathbb{R}^n_+$. To exploit its geometry, we describe the capacity region $\mathcal{C}$ as follows

$$\mathcal{C} = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \ \langle \boldsymbol{c}^{(\ell)}, \boldsymbol{x} \rangle \leq b^{(\ell)} \ , \ \ell = 1, \ldots, L \right\}, \tag{4}$$

where L is the minimum number of hyperplanes needed to describe $\mathcal{C}$. Without loss of generality, we assume $\|\boldsymbol{c}^{(\ell)}\| = 1$, $\boldsymbol{c}^{(\ell)} \geq 0$ and $b^{(\ell)} > 0$ for all $\ell \in [L]$. For each $\ell \in [L]$, let $\mathcal{F}^{(\ell)}$ be the ℓ^{th} facet of $\mathcal{C}$, i.e. $\mathcal{F}^{(\ell)} = \left\{ \boldsymbol{x} \in \mathcal{C} : \ \langle \boldsymbol{c}^{(\ell)}, \boldsymbol{x} \rangle = b^{(\ell)} \right\}$.

By definition of the capacity region and since we are using MaxWeight algorithm to select schedules, it is clear that $\boldsymbol{s}(k)$ does not necessarily belong to the capacity region $\mathcal{C}$. To overcome this difficulty, we use the following lemma.

Lemma 1. *Consider a generalized switch operating under MaxWeight as described above. Then,*

$$\mathbb{E}\left[\langle \boldsymbol{q}(k), \boldsymbol{s}(k) \rangle \,|\, \boldsymbol{q}(k) = \boldsymbol{q} \right] = \max_{\boldsymbol{x} \in \mathcal{C}} \langle \boldsymbol{q}, \boldsymbol{x} \rangle.$$

For each $\ell \in [L]$ and $j \in \mathcal{J}$ define the *maximum* $\boldsymbol{c}^{(\ell)}$*-weighted service rate available when channel state is* j as $b^{(j,\ell)} = \max_{\boldsymbol{x} \in \mathcal{S}^{(j)}} \langle \boldsymbol{c}^{(\ell)}, \boldsymbol{x} \rangle$. Observe that, if the channel state is fixed (i.e. if $\mathcal{J}$ only has one element), then $b^{(j,\ell)} = b^{(\ell)}$. For each $\ell \in [L]$, let $\{B_\ell(k) : \ k \geq 1\}$ be an i.i.d. process, independent of the queue lengths process, that satisfies $\mathbb{P}\left[B_\ell(1) = b^{(j,\ell)}\right] = \psi_j$ and let $\sigma^2_{B_\ell} = \text{Var}[B_\ell(1)]$. The processes $\{B_\ell(k) : \ k \geq 1\}$ are independent across ℓ.

Now we describe how we model heavy-traffic in this paper. We fix a vector $\boldsymbol{\nu}$ in the boundary of $\mathcal{C}$ and we consider a set of generalized switches operating under MaxWeight as described above, parametrized by $\epsilon \in (0, 1)$. The heavy-traffic limit is the limit as $\epsilon \downarrow 0$ and, as ϵ gets small, the vector of mean arrival rates approaches $\boldsymbol{\nu}$.

Formally, we consider a set of generalized switches operating under MaxWeight, parametrized by $\epsilon \in (0, 1)$ in the following way. We let $\boldsymbol{q}^{(\epsilon)}(k)$, $\boldsymbol{a}^{(\epsilon)}(k)$, $\boldsymbol{s}^{(\epsilon)}(k)$ and $\boldsymbol{u}^{(\epsilon)}(k)$ be the vectors of queue lengths, arrivals, offered services and unused services, respectively, in time slot k, in the system parametrized by ϵ. The parametrization is such that the vector of mean arrival rates be $\boldsymbol{\lambda}^{(\epsilon)} = \mathbb{E}\left[\boldsymbol{a}^{(\epsilon)}(1)\right] = (1-\epsilon)\boldsymbol{\nu}$. Therefore, $\boldsymbol{\lambda}^{(\epsilon)}$ belongs to the interior of $\mathcal{C}$ for each ϵ and, as $\epsilon \downarrow 0$, $\boldsymbol{\lambda}^{(\epsilon)}$ approaches the boundary of the capacity region.

Heavy-traffic analysis of the generalized switch was studied in [1] when the vector $\boldsymbol{\nu}$ is in the interior of a facet of the capacity region $\mathcal{C}$. In that case, SSC occurs into a one-dimensional subspace and the CRP condition is satisfied. In this paper, we focus on the case when the vector $\boldsymbol{\nu}$ lives in the intersection of facets. We let $P \subset [L]$ be the maximal set of indexes of the facets that intersect at $\boldsymbol{\nu}$. Then $\boldsymbol{\nu} \in \bigcap_{\ell \in P} \mathcal{F}^{(\ell)}$. We say P is maximal in the sense that if $\tilde{\ell} \in [L] \setminus P$, then $\bigcap_{\ell \in P \cup \{\tilde{\ell}\}} \mathcal{F}^{(\ell)} = \emptyset$.

In the next section we present our results and a sketch of our proofs.

3. ASYMPTOTICALLY TIGHT BOUNDS

In this section we apply the drift method to compute bounds on a linear combination of the queue lengths, that are tight in heavy-traffic. We start with SSC. Define

$$\mathcal{H} = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \ \boldsymbol{x} = \sum_{\ell \in P} \alpha_\ell \boldsymbol{c}^{(\ell)} \ , \ \alpha_\ell \in \mathbb{R} \quad \forall \ell \in P \right\}$$

and $\mathcal{K} = \mathcal{H} \cap \mathbb{R}_+$. Observe that $\mathcal{K}$ is a cone on $\mathbb{R}_+$ and that $\mathcal{H}$ is the subspace generated by the cone $\mathcal{K}$. For each $\epsilon \in (0,1)$, let $\boldsymbol{q}^{(\epsilon)}_{\|\mathcal{K}}(k)$ be the projection of $\boldsymbol{q}^{(\epsilon)}(k)$ on $\mathcal{K}$ and $\boldsymbol{q}^{(\epsilon)}_{\perp\mathcal{K}}(k) = \boldsymbol{q}^{(\epsilon)}(k) - \boldsymbol{q}^{(\epsilon)}_{\|\mathcal{K}}(k)$. Similarly, let $\boldsymbol{q}^{(\epsilon)}_{\|\mathcal{H}}(k)$ be the projection of $\boldsymbol{q}^{(\epsilon)}(k)$ on $\mathcal{H}$ and $\boldsymbol{q}^{(\epsilon)}_{\perp\mathcal{H}}(k) = \boldsymbol{q}^{(\epsilon)}(k) - \boldsymbol{q}^{(\epsilon)}_{\|\mathcal{H}}(k)$. The queueing system is stable for each $\epsilon \in (0,1)$, so we define $\overline{\boldsymbol{q}}$, $\overline{\boldsymbol{q}}_{\|\mathcal{K}}$, $\overline{\boldsymbol{q}}_{\perp\mathcal{K}}$, $\overline{\boldsymbol{q}}_{\|\mathcal{H}}$ and $\overline{\boldsymbol{q}}_{\perp\mathcal{H}}$ to be steady-state vectors to which $\boldsymbol{q}(k)$, $\boldsymbol{q}_{\|\mathcal{K}}(k)$, $\boldsymbol{q}_{\perp\mathcal{K}}(k)$, $\boldsymbol{q}_{\|\mathcal{H}}(k)$ and $\boldsymbol{q}_{\perp\mathcal{H}}(k)$ converge in distribution, respectively. In the next proposition we state SSC formally.

Proposition 1. *Consider a generalized switch operating under MaxWeight, parametrized by ϵ as described in Section 2. Then, for each $m = 1, 2, \dots$ there exists a finite constant M_m such that $\mathbb{E}\left[\|\overline{\boldsymbol{q}}^{(\epsilon)}_{\perp\mathcal{H}}\|^m\right] \leq \mathbb{E}\left[\|\overline{\boldsymbol{q}}^{(\epsilon)}_{\perp\mathcal{K}}\|^m\right] \leq M_m$.*

SSC is a consequence of Proposition 1 for the following reason. As $\epsilon \downarrow 0$, $\|\overline{\boldsymbol{q}}\|$ goes to infinity (this can be seen from the proof of Theorem 1). Therefore, Proposition 1 implies that as ϵ gets small, we can approximate $\overline{\boldsymbol{q}} \approx \overline{\boldsymbol{q}}_{\|\mathcal{K}}$ because all the moments of $\|\overline{\boldsymbol{q}}_{\perp\mathcal{K}}\|$ are bounded above.

Proof sketch of Proposition 1. The first inequality holds because $\mathcal{K} \subset \mathcal{H}$ by definition. The proof of the second inequality is based on Lemma 1 in [1], which is a Foster-Lyapunov type of argument. Basically, we prove that $\mathbb{E}\left[\|\boldsymbol{q}_{\perp\mathcal{K}}(k+1)\| - \|\boldsymbol{q}_{\perp\mathcal{K}}(k)\| \mid \boldsymbol{q}(k) = \boldsymbol{q}\right]$ is bounded above by a negative number if $\|\boldsymbol{q}_{\perp\mathcal{K}}\|$ is large and that $\left|\|\boldsymbol{q}_{\perp\mathcal{K}}(k+1)\| - \|\boldsymbol{q}_{\perp\mathcal{K}}(k)\|\right| \mathbb{1}_{\{\boldsymbol{q}(k)=\boldsymbol{q}\}}$ is bounded with probability one for all $\boldsymbol{q}$. □

The main contribution of our work is the following theorem.

Theorem 1. *Consider a set of generalized switches operating under MaxWeight, indexed by the heavy-traffic parameter $\epsilon \in (0,1)$ as described in Section 2. Then,*

$$\left| \mathbb{E}\left[\langle \overline{\boldsymbol{q}}^{(\epsilon)}, \boldsymbol{\nu} \rangle\right] - \frac{1}{2\epsilon}\left(\sum_{r,t=1}^{n} h_{r,t}^2 \sigma_{a_t}^2 + \sum_{r=1}^{n} \sum_{\ell \in P} \widetilde{h}_{r,\ell}^2 \sigma_{B_\ell}^2 \right) \right| \leq K(\epsilon),$$

where $\epsilon K(\epsilon)$ converges to 0 as $\epsilon \downarrow 0$; for each $r, t \in \{1, \dots, n\}$ $h_{r,t}$ is the $(r,t)^{th}$ element of the projection matrix on $\mathcal{H}$, $H \triangleq C(C^T C)^{-1} C^T$ with C a matrix with columns $\left[\boldsymbol{c}^{(\ell)}\right]_{\ell \in P}$; and for each $r \in \{1, \dots, n\}$ and $\ell \in P$, $\widetilde{h}_{r,\ell}$ is the $(r,\ell)^{th}$ element of $\widetilde{H} \triangleq C(C^T C)^{-1}$.

Thus, in the heavy-traffic limit as $\epsilon \downarrow 0$, we have

$$\lim_{\epsilon \downarrow 0} \epsilon\, \mathbb{E}\left[\langle \overline{\boldsymbol{q}}^{(\epsilon)}, \boldsymbol{\nu} \rangle\right] = \frac{1}{2}\left(\sum_{r,t=1}^{n} h_{r,t}^2 \sigma_{a_t}^2 + \sum_{r=1}^{n} \sum_{\ell \in P} \widetilde{h}_{r,\ell}^2 \sigma_{B_\ell}^2 \right)$$

Proof sketch of Theorem 1. For ease of exposition, in this proof we omit the dependence on ϵ of the variables and we add a line on top of them when we work in steady-state.

We set to zero the drift of $V(\boldsymbol{q}) = \left\|\boldsymbol{q}_{\|\mathcal{H}}\right\|^2$ and we obtain

$$\mathcal{T}_1 = \mathcal{T}_2 - \mathcal{T}_3 + \mathcal{T}_4,$$

where

$$\mathcal{T}_1 \triangleq 2\mathbb{E}\left[\langle \overline{\boldsymbol{q}}_{\|\mathcal{H}}, \overline{\boldsymbol{s}}_{\|\mathcal{H}} - \overline{\boldsymbol{a}}_{\|\mathcal{H}} \rangle\right] \ , \ \mathcal{T}_2 \triangleq \mathbb{E}\left[\left\|\overline{\boldsymbol{a}}_{\|\mathcal{H}} - \overline{\boldsymbol{s}}_{\|\mathcal{H}}\right\|^2\right] \ ,$$
$$\mathcal{T}_3 \triangleq \mathbb{E}\left[\left\|\overline{\boldsymbol{u}}_{\|\mathcal{H}}\right\|^2\right] \text{ and } \mathcal{T}_4 \triangleq 2\mathbb{E}\left[\langle \overline{\boldsymbol{q}}^+_{\|\mathcal{H}}, \overline{\boldsymbol{u}}_{\|\mathcal{H}} \rangle\right].$$

Using the geometry of the subspace where SSC occurs and Lemma 1, we obtain

$$\mathcal{T}_1 = 2\epsilon\mathbb{E}\left[\langle \overline{\boldsymbol{q}}_{\|\mathcal{H}}, \boldsymbol{\nu} \rangle\right] + O(\sqrt{\epsilon})$$
$$\mathcal{T}_2 = \sum_{r,t=1}^{n} h_{r,t}^2 \left(\sigma_{a_t}^{(\epsilon)}\right)^2 + \sum_{r=1}^{n} \sum_{\ell \in P} \widetilde{h}_{r,\ell}^2 \sigma_{B_\ell}^2 + \|\boldsymbol{\nu}\|^2 \epsilon^2 + O(\epsilon)$$
$$|\mathcal{T}_3| \text{ is } O(\epsilon) \quad \text{and} \quad |\mathcal{T}_4| \text{ is } O(\epsilon^{\frac{1}{8}}),$$

which completes the proof. Observe that the bounds arise from the absolute values in the terms $\mathcal{T}_3$ and $\mathcal{T}_4$. □

4. CONCLUSION

In this paper we used the drift method as developed in [1] to compute asymptotically tight bounds for the generalized switch operating under MaxWeight when SSC occurs into a multidimensional space. Our results generalize the work in [1, 4], where the drift method was used to study heavy-traffic performance of queueing systems that are particular cases of the generalized switch.

5. REFERENCES

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