Ergodicity of Time Reversal Process of Stochastic Consensus Formation and Its Application

Shigeo Shioda and Kenta Takehara
Graduate School of Science and Engineering, Chiba University
1-33 Yayoi, Inage, Chiba 263-8522, Japan
shioda@faculty.chiba-u.jp, take130@icloud.com

ABSTRACT

The consensus reached in stochastic consensus formation is a random variable whose distribution is generally difficult to determine analytically. We show that the time reversal process for the stochastic consensus formation process is ergodic. This fact allows us to numerically obtain the distribution for the consensus by observing the time reversal process for consensus formation for a fixed sample path.

1. INTRODUCTION

Consensus formation is a problem in which agents with initially different opinions mutually exchange and update their opinions using a distributed algorithm to achieve a consensus. This problem has appeared in various contexts [3], including distributed computation, load balancing in computer networks, distributed data fusion or clock synchronization in sensor networks, coordinate control of mobile agents, and opinion formation in social networks.

In some consensus formation algorithms, agents are stochastically selected for exchanging and updating their opinions at each opinion update epoch. Such consensus formation algorithms can be referred to as stochastic consensus formation algorithms. The consensus obtained in stochastic consensus formation is not a constant, but a random variable [4, 2]. It is desirable to know the distribution of the obtained consensus *a priori* when applying a consensus formation algorithm to engineering problems such as sensor data fusion and coordinate control of mobile agents; however, it is generally difficult to determine the distribution of the reached consensus analytically. To numerically obtain the distribution of consensus results, it is necessary to carry out a large number of simulations with different seeds for random number generation.

In this paper, we study the time reversal process for stochastic consensus formation. In particular, we show that this time reversal process is ergodic. This fact allows us to numerically obtain the distribution of the consensus by observing the time reversal process for consensus formation in only one simulation.

The remainder of the paper is organized as follows. In Sec. 2, we explain the mathematical model for stochastic consensus formation. Then, in Sec. 3.1, we show the ergodicity of the time reversal process and, using a few numerical examples, we explain how it can be used to obtain the distribution of the consensus.

MAMA 2022 Numbai, India

2. STOCHASTIC CONSENSUS FORMATION

2.1 Model

We consider N agents interacting over a directed graph. The agents are numbered from 1 to N. Each agent has its own opinion, which is expressed as a real number. The agents exchange and update their opinions at discrete times $t \in \mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, \dots\}$. Let $x_n(t) \in \mathbb{R}$ denote the opinion of agent n at time t and let $\boldsymbol{x}(t)$ denote a row vector whose elements are equal to the opinions of agents at time t; that is, $\boldsymbol{x}(t) = (x_1(t), \cdots, x_N(t))$. At time t, $\boldsymbol{x}(t)$ is updated to $\boldsymbol{x}(t+1)$ using the following equation:

$$\boldsymbol{x}(t+1)^\mathsf{T} = Q^{(e_t)}\boldsymbol{x}(t)^\mathsf{T}, \tag{1}$$

where $Q^{(e_t)}$ is a non-negative matrix, referred to as the opinion-update matrix in this paper, that expresses the opinion updates at time t. We assume that there are K types of opinion-update matrix. $e_t \in \{1, \dots, K\}$ denotes a random variable that expresses the type of opinion-update matrix used at time t. $e_0, e_1, \dots$ are independent and identically distributed. Note that in stochastic consensus formation, the opinion-update matrix is not unique and different matrices are used at each opinion update epoch. We assume that $Q^{(k)} = \{q_{ij}^{(k)}\}$ $(k = 1, \dots, K)$ is a stochastic matrix; that is, the sum of elements in each row is equal to 1. It follows from (1) that

$$E[\boldsymbol{x}(t+1)|\boldsymbol{x}(t)] = Q\boldsymbol{x}(t)^\mathsf{T}, \quad Q \stackrel{\text{def}}{=} \sum_{k=1}^{N} p_k Q^{(k)}, \tag{2}$$

where $p_k = P(e_t = k)$. Note that Q is also a stochastic matrix. If a random variable x_c exists and satisfies

$$\lim_{t\to\infty} \boldsymbol{x}(t) = x_c\mathbf{1},\ a.s., \quad \mathbf{1} \stackrel{\text{def}}{=} \{1, \dots, 1\},$$

then we say that a consensus is reached and that it is equal to x_c. It has been proved [2] that a consensus is reached if Q is irreducible[1]. We define $\bar{X}(t) \stackrel{\text{def}}{=} \boldsymbol{\pi}\boldsymbol{x}(t)^\mathsf{T}$, where $\boldsymbol{\pi}$ is the left eigenvector of Q corresponding to the largest eigenvalue; $\boldsymbol{\pi}$ is normalized so that the sum of elements is equal to 1. If a consensus is reached, $\lim_{t\to\infty} \bar{X}(t) = x_c$ holds with probability 1 and $E[\bar{X}(t)] = x_c$ [2].

[1] Although [2] considered broadcast-based consensus formation, its proof is applicable to general stochastic consensus formation.

2.2 Example of Stochastic Consensus Formation Algorithm

Broadcast-based consensus formation is a type of stochastic consensus formation where one of the agents is selected to broadcast its opinion to its neighbors. Each agent that has received its neighbor's opinion calculates the weighted average of its opinion and the received opinion, and replaces its opinion with the calculation result. Let $Q^{(k)}$ denote the opinion-update matrix when agent k broadcasts its opinion. The non-diagonal elements of $Q^{(k)}$, $q_{ij}^{(k)} (i \neq j)$, are expressed as

$$q_{ij}^{(k)} = \begin{cases} \frac{a_{ki} r_k}{\epsilon r_i + r_k}, & j = k \\ 0, & \text{otherwise} \end{cases}$$

where $r_i (\geq 0)$ denotes a parameter, called the influence parameter, that indicates the strength of the influence on the opinions of other agents, and a_{ij} expresses the existence of a directed link from agent i to agent j. If a directed link exists, then $a_{ij} = 1$; otherwise, $a_{ij} = 0$. Parameter ϵ indicates the strength of interaction between agents. A smaller value of ϵ indicates stronger interaction between agents.

The gossip algorithm is a type of stochastic consensus formation algorithm where a pair of agents connected via a bi-directional link is stochastically selected to exchange and update their opinions at each opinion update time. Assume that agents k_1 and k_2 are selected for updating the opinions in the kth type of opinion update. The non-diagonal elements of $Q^{(k)}$ are all equal to zero, except for $q_{k_1 k_2}^{(k)}$ and $q_{k_2 k_1}^{(k)}$, which are respectively given as

$$q_{k_1 k_2}^{(k)} = \frac{r_{k_2}}{\epsilon r_{k_1} + r_{k_2}}, \quad q_{k_2 k_1}^{(k)} = \frac{r_{k_1}}{r_{k_1} + \epsilon r_{k_2}}.$$

3. ERGODICITY OF TIME REVERSAL PROCESS OF CONSENSUS FORMATION

3.1 Ergodicity of Time Reversal Process

Using (1), we can express $\boldsymbol{x}(t)$ as follows.

$$\boldsymbol{x}(t)^\top = Q^{(e_{t-1})} \times \cdots \times Q^{(e_0)} \boldsymbol{x}(0)^\top.$$

Define

$$\boldsymbol{y}(t)^\top \stackrel{\text{def}}{=} Q^{(e_0)} \times \cdots \times Q^{(e_{t-1})} \boldsymbol{x}(0)^\top, \quad \bar{Y}(t) \stackrel{\text{def}}{=} \boldsymbol{\pi} \boldsymbol{y}(t)^\top,$$

where $\boldsymbol{y}(t)$ can be regarded as a time reversal process for $\boldsymbol{x}(t)$. Because $e_0, e_1, \ldots$ are independent and identically distributed, $\bar{Y}(t)$ and $\bar{X}(t)$ have the same distribution.

In the following, we assume that e_t $(t \in \mathbb{N})$ is a random variable in probability space $(\Omega, \mathscr{F}, P)$ $(e_t : \Omega \to \{1, \ldots, N\})$. On $(\Omega, \mathscr{F}, P)$, a measurable map $\theta : \Omega \to \Omega$ and a random variable e exist; they satisfy

$$e_t(\omega) = e(\theta^t \omega), \quad (e_t = e \circ \theta^t).$$

For each $\omega \in \Omega$, there exists $\omega_r \in \Omega$ that satisfies

$$\forall t \in \mathbb{Z}, \quad e_t(\omega) = e_{-t}(\omega_r),$$

where $\mathbb{Z}$ denotes the set of whole integers. Let $\xi : \Omega \to \Omega$ be a measurable map that relates ω and ω_r such that

$$\forall \omega \in \Omega, \quad \omega_r = \xi \omega.$$

Assume that θ and ξ are measure-preserving; that is, $P(\theta A) = P(A)$ and $P(\xi A) = P(A)$ for all $A \in \mathscr{F}$. We also assume that θ is ergodic. Because

$$e_t = e_{-t} \circ \xi = e \circ \theta^{-t} \xi = e_s \circ \theta^{-s-t} \xi,$$

it follows that

$$\begin{aligned} \bar{Y}(t) &= \boldsymbol{\pi} Q^{(e_0)} \times \cdots \times Q^{(e_{t-1})} \boldsymbol{x}(0)^\top \\ &= \boldsymbol{\pi} Q^{(e_{t-1} \circ \theta^{1-t} \xi)} \times \cdots \times Q^{(e_0 \circ \theta^{1-t} \xi)} \boldsymbol{x}(0)^\top \\ &= \bar{X}(t) \circ \theta^{1-t} \xi. \end{aligned}$$

Thus, we obtain

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} \bar{Y}(t) = \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} \bar{X}(t) \circ \theta^{1-t} \xi. \tag{3}$$

Because $\bar{X}(t)$ converges to x_c with probability 1 as $t \to \infty$, we can expect that the right-hand side of the above equation will be equal to $E[x_c]$. In fact, we can prove the following theorem. (Because of space limitations, the proof is omitted.)

THEOREM 1.

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} \bar{Y}(t) = E[x_c], \quad a.s.$$

Because $E[\bar{Y}(t)] = E[\bar{X}(t)] = E[x_c]$, Theorem 1 means that time reversal process $\bar{Y}(t)$ is ergodic. By the same arguments, we also obtain

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} 1_{(\bar{Y}(t) \leq a)} = P(x_c \leq a). \tag{4}$$

3.2 Iterated Random Functions

We can express $\bar{Y}(t)$ in the following form:

$$\bar{Y}(t) = \boldsymbol{x}(0) \tilde{\boldsymbol{y}}(t)^\top, \quad \tilde{\boldsymbol{y}}(t)^\top \stackrel{\text{def}}{=} {Q^{(e_{t-1})}}^\top \times \cdots \times {Q^{(e_0)}}^\top \boldsymbol{\pi}^\top, \tag{5}$$

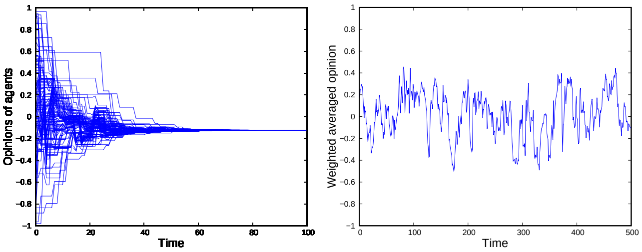
where $\tilde{\boldsymbol{y}}(t)$ can be viewed as a process constructed by the following iterated random functions:

$$\begin{aligned} &\tilde{\boldsymbol{y}}(1)^\top = f_{e_0}(\boldsymbol{\pi}^\top), \ \tilde{\boldsymbol{y}}(2)^\top = f_{e_1}(\tilde{\boldsymbol{y}}(1)^\top) = (f_{e_1} \circ f_{e_0})(\boldsymbol{\pi}^\top), \\ &\tilde{\boldsymbol{y}}(t)^\top = f_{e_{t-1}}(\tilde{\boldsymbol{y}}(t-1)^\top) = (f_{e_{t-1}} \circ \cdots \circ f_{e_0})(\boldsymbol{\pi}^\top), \end{aligned}$$

where $f_{e_n} \stackrel{\text{def}}{=} {Q^{(e_n)}}^\top$. We also see that

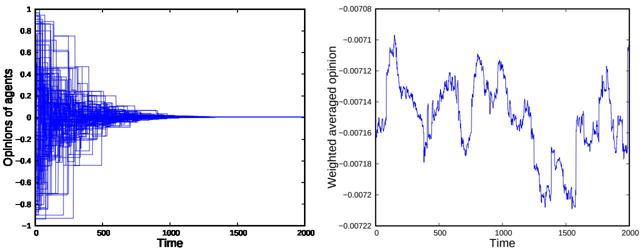
$$\bar{X}(t) = \boldsymbol{x}(0) \tilde{\boldsymbol{x}}(t)^\top, \quad \tilde{\boldsymbol{x}}(t)^\top \stackrel{\text{def}}{=} (f_{e_0} \circ \cdots \circ f_{e_{t-1}})(\boldsymbol{\pi}^\top),$$

and thus $\tilde{\boldsymbol{x}}(t)$ is the reverse process of $\tilde{\boldsymbol{y}}(t)$. It was shown in [1] that, under some regularity conditions (Theorem 1.1 in [1]), a forward process constructed by iterated random functions has a unique stationary distribution and moves ergodically, whereas the corresponding reverse process converges almost surely to a limit. Based on the discussion in Sec. 3.1, consensus formation and its time reversal process correspond to the pair of forward and reverse processes discussed in [1]. The discussion in Sec. 3.1 also shows that the irreducibility of Q, a condition that ensures that consensus is reached almost surely, is a sufficient condition for satisfying the regularity conditions of Theorem 1.1 in [1] in the problem of consensus formation.



(a) Consensus formation (b) Time reversal process

Figure 1: Consensus formation and its time reverse (broadcast)



(a) Consensus formation (b) Time reversal process

Figure 2: Consensus formation and its time reversal (gossip)

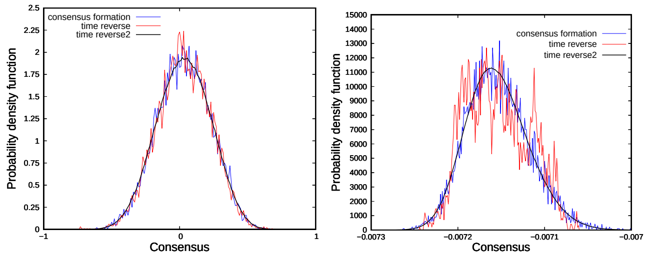
3.3 Applications

Equation (5) indicates that the time reversal process $\bar{Y}(t)$ can be constructed by observing $e_0, e_1, \ldots$ in normal time order. If $\bar{Y}(t)$ is observed for some duration with a fixed sample path, the distribution of the consensus can be obtained through (4). Suppose that we want M samples of the consensus to know its distribution and it takes T updates of opinions to almost reach a consensus. Then, we have to conduct M different T-time-unit simulations when an update of opinions takes one time unit; therefore, the simulation runs for a total of TM time units. The distribution of the consensus can also be obtained by observing the time reversal process $\bar{Y}(t)$ from $t = \tilde{T} + 1$ to $t = \tilde{T} + \tilde{M}$ in one simulation, where $\tilde{T}$ is the mixing time and $\tilde{M}$ should be somewhat larger than M because $Y(\tilde{T}+1), Y(\tilde{T}+2), \ldots$ are not independent but correlated. If $T \approx \tilde{T} \ll \tilde{M} \ll MT$, however, the latter requires a much shorter time than the former because $\tilde{M} + \tilde{T} \ll MT$.

Figure 1(a) shows the time change of the opinions of agents in broadcast-based consensus formation for 100 agents connected by a directed network, which was generated by establishing a directed link between each agent pair with probability 0.3. The initial opinions of agents were given according to the uniform distribution between -1 and 1. Parameter ϵ was set at 1. The influence parameter $\{r_i\}_{i=1}^{N}$ was given as

$$r_i = \max\{N(1, 0.2), 0\},$$

where $N(1, 0.2)$ denotes a random variable that follows a Gaussian distribution with mean 1 and variance 0.2. As shown in Fig. 1(a), the opinions of agents quickly reached a consensus. Figure 1(b) shows the time change of $\bar{Y}(t)$ obtained in the same setting. As shown, $\bar{Y}(t)$ did not converge to a fixed real number. Figures 2(a) and 2(b) respectively show the time change of the opinions of agents and the time change of $\bar{Y}(t)$ in gossip-based consensus formation when $N = 100$ and $\epsilon = 4$. The initial opinions and the influence parameters for agents were the same as those in the broadcast-based consensus formation experiments. The results in Figs. 2(a) and 2(b) are similar to those in Figs. 1(a) and 1(b), but the time required to reach consensus is much longer and the range of $\bar{Y}(t)$ values is much narrower than those for broadcast-based consensus formation.



(a) Broadcast (b) Gossip

Figure 3: Comparison of probability density functions for consensus obtained by three methods

Figure 3(a) shows three probability density functions for the consensus; the red curve is the one obtained by conducting $10^4 (= M)$ simulations with different random seeds when $T = 200$, the blue curve is the one obtained by observing the time reversal process in one simulation when $\tilde{T} = 200$ and $\tilde{M} = M (= 10^4)$, and the black curve is the one obtained by running the time reversal process when $\tilde{T} = 200$ and $\tilde{M} = MT - \tilde{T} (\approx 10^6)$. Note that for the red and black curves, the simulation ran for the same total length of time (2×10^6 time units), which was longer than that required for the blue curve ($10^4 + 200$ time units). The black curve is much smoother than the blue curve. That is, for a given length of time, observing the time reversal process is much more efficient for obtaining the distribution. The red and blue curves are almost equally smooth. Figure 3(b) shows three probability density functions for the consensus in gossip-based consensus formation. We set $T = \tilde{T} = 8000$ and $M = \tilde{M} = 10^5$. The black curve is much smoother than the red and blue curves. The red curve is less smooth than the blue curve because $\bar{Y}(t)$ is a rather strongly correlated process, as shown in Fig. 2(b).

Acknowledgments

This work was supported by the Japan Society for the Promotion of Science (JSPS) KAKENHI (grant number JP20K21783).

References

[1] P. Diaconis and D. Freedman. Iterated random functions. *SIAM review*, 41(1):45–76, 1999.

[2] D. Katoh and S. Shioda. Probability laws of consensus in a broadcast-based consensus-forming algorithm. *Stochastic Models*, 38(1):91–115, 2022.

[3] R. Olfati-Saber, J. A. Fax, and R. M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, 2007.

[4] S. Shioda. Distribution of consensus in a broadcasting-based consensus-forming algorithm. *ACM SIGMETRICS Performance Evaluation Review*, 48(3):91–96, 2020.