On the performance of Car-Sharing Systems

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1. INTRODUCTION

Over the past decade, vehicle-sharing systems have appeared as a new answer to mobility challenges, like reducing congestion, pollution or travel time for numerous cities.

For bike-sharing systems, users make one-way trips. The usage is the following: Users pick a bike at a station if one is available. Otherwise usually they leave the system and take another mode of transportation. After the trip, they return the bike at another station if there is an available space. Otherwise, they have to find a neighboring station to return the bike. In car-sharing systems, users have the ability to reserve, for example the parking space, avoiding the problem of looking for an available space at destination. Again, they can only do this if there are cars and spaces available. The lack of resources, both vehicles and parking slots, is one of the major issues for operators to maintain the reliability of the service (see [6], [1]).

In this paper we analyze a simple homogeneous model for car-sharing systems with one-way trips where users reserve the parking space just as the car is picked up. As far as we know, it is the first stochastic model of a large-scale vehicle-sharing system with reservation.

Model Description. The system is a set of N stations with capacity K with M cars. Users arrive at each station according to a Poisson process with rate λ. An arriving user at station i chooses a destination j at random. If there is no car available in station i *or* if there is no available parking space in station j, the user leaves the system. Otherwise, he picks up a car at station i and simultaneously makes a reservation at station j. Then, the trip between station i and station j takes an exponentially distributed time with mean $1/\mu$. The user returns his car at station j and leaves the system.

In this paper, we focus on this homogeneous model. But we can extend the result to a heterogeneous model consisting in a finite number of clusters with capacity K_i and arrival rate λ_i for a station in cluster i and probability p_i of choosing a destination in cluster i, instead of $1/N$, as in [4].

Large Scale Behavior For this model, the state process of the numbers of cars and reserved parking spaces is an irreducible Markov process on a finite state space with no explicit expression for the unique invariant measure. Thus the aim is to investigate the large scale behavior of the system. This means asymptotics when the number of cars and stations are large together, i.e. that their ratio tends to a constant. Our first result (Theorem 1) is the convergence when the system gets large of the distribution of a station state to the distribution of a non linear inhomogeneous Markov process. This distribution is given by a differential equation called Fokker-Planck equation.

Our second main result (Theorem 2) establishes the existence and uniqueness of the equilibrium point of this ODE. The proof uses a monotonicity argument similar as in [3], but here with intricate calculations. We can fully characterize it using probabilistic interpretation.

Our goal is to study the system performance in terms of large scale stationary proportion of empty and full stations, in particular the fleet size influence. For the optimal fleet size, we give asymptotics for performance in light and heavy traffic. We prove that, in light traffic case, reservation has little impact on performance, unlike the heavy traffic case.

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2. STOCHASTIC MODEL

2.1 State Process

Let us define $\chi = \{(k,l) \in \mathbb{N}^2, k+l \leq K\}$ and let $(X^N(t)) = (R_i^N(t)), V_i^N(t), 1 \leq i \leq N)$ where

- $R_i^N(t)$ is the number of reserved parking spaces in station i at time t,
- $V_i^N(t)$ is the number of cars in station i at time t.

$(X^N(t))$ is a Markov process at time t on

$$\mathcal{S}^N = \{x = (r_i, v_i)_{1 \leq i \leq N}, \forall i, (r_i, v_i) \in \chi, \sum_{i=1}^{N} r_i + v_i = M\}$$

with jump matrix

$$\begin{cases} Q^N(x, x + e_j - f_i) = \lambda/N \, 1_{\{v_i > 0,\ r_j + v_j < K\}} \\ Q^N(x, x - e_j + f_j) = \mu \, r_j \end{cases}$$

where $(e_i, f_i, 1 \leq i \leq N)$ is the canonical basis of $\mathbb{R}^{2K}$. Indeed, the transitions from $x \in \mathcal{S}^N$ are the following:

- **Cars picked up**. A user arrives at station i at rate λ and takes a car if $v_i > 0$. At the same time he reserves in station j with probability $1/N$ if $r_j + v_j < K$. Thus, the transition rate is λ/N if $v_i > 0, r_j + v_j < K$. Due to car picked up, v_i decreases by 1. Due to reservation of the parking space, r_j increases by 1.
- **Cars returned**. When a car arrives at its reserved parking space in station j, r_j decreases by 1 and v_j increases by 1. As the trips are exponentially distributed with mean $1/\mu$, this transition occurs at rate $\mu \, r_j$.

As $(X^N(t))$ is irreducible and on finite set $\mathcal{S}^N$, it is ergodic with a unique invariant measure, which is analytically untractable. Thus we investigate its limit as N tends to $+\infty$ via a mean-field approach.

2.2 Dynamical System

THEOREM 1 (MEAN-FIELD CONVERGENCE THEOREM). *If $(R_i^N(t)), V_i^N(t)$ has an initial distribution converging to some distribution y_0, then it converges in distribution to some non-homogeneous Markov process $(R(t), V(t))$ whose $\mathcal{Q}$-matrix $Q(t)$ is given by*

$$\begin{cases} Q(t)((k,l),(k+1,l)) & = \lambda\,\mathbb{P}(V(t)>0)1_{\{k+l<K\}}, \\ Q(t)((k,l),(k-1,l+1)) & = \mu\, l, \\ Q(t)((k,l),(k,l-1)) & = \lambda\,\mathbb{P}(R(t)+V(t)<K) \end{cases}$$

with $\mathbb{E}\left(R(t)+V(t)\right) = s$.

Remark. It means that $(R(t), V(t))$ is the joint number of customers in a tandem of two queues with total capacity K and arrival rate $\lambda(1-y_{\cdot,0}(t))$,

- a $M/M/\infty$ queue with service rate μ,
- a $M/M/1$ queue with service rate $\lambda(1-y_S(t)$

where $y(t)$ is its distribution, valued in

$$\mathcal{Y} = \{y \in \mathcal{P}(\chi), \sum_{(k,l)\in\chi} (k+l)y_{k,l} = s\}.$$

and where $y_S(t) = \sum_{k+l=K} y_{k,l}(t)$ and $y_{\cdot,0}(t) = \sum_{l=0}^{K-k} y_{l,0}(t)$ are respectively the limiting proportion of saturated stations at t and the limiting proportion of stations with no cars at t.

The distribution $y(t)$ of $(R(t), V(t))$ is characterized as the solution of the ODE

$$\dot{y}(t) = y(t)L_{y(t)} \tag{1}$$

with fixed initial condition and $L_{y(t)} = Q(t)$.

PROOF. The proof is technical and uses the empirical measure process. See [2] for details. □

We concentrate on the invariant distribution of $(X(t))$. Due to the non-homogeneity of Markov process $(X(t))$, the uniqueness property is not clear. It is the object of Theorem 2. Then Section 4 is devoted to the influence of fleet size on performance.

3. STEADY-STATE BEHAVIOR

3.1 Invariant measure

Recall that the Markov process with generator L_y on χ previously introduced in the remark of Theorem 1 is the joint number of customers in two queues in tandem (see Figure 1): a $M/M/\infty$ queue *for reservations*, with arrival rate $\lambda(1-y_{\cdot,0})$ and service rate μ, and a $M/M/1$ queue *for cars*, where customers come from the former queue, with service rate $\lambda(1-y_S)$. The system is a loss system with total capacity K. The key point is the explicit form for the invariant measure $\pi(y)$ of the Markov process.

Indeed, let ρ_R be the arrival-to-service rate ratio of the $M/M/\infty$ queue, and ρ_V of the $M/M/1$ queue, defined as

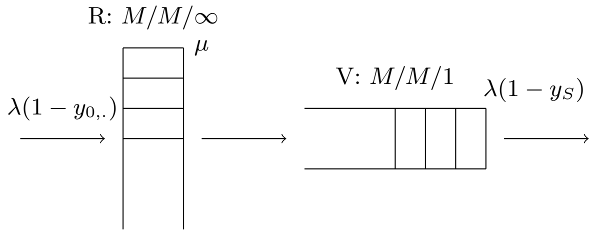

Figure 1: The state of a station as a tandem of two queues with overall capacity K.

$$\rho_R(y) = \frac{\lambda}{\mu}(1-y_{\cdot,0}), \;\; \rho_V(y) = \frac{1-y_{\cdot,0}}{1-y_S}. \tag{2}$$

With these notations, (see [5] for example) the invariant probability measure $\pi(y)$ associated to L_y has product form given by, with a slight abuse of notations, for $(k,l)\in\chi$,

$$\pi_{k,l}(\rho_R(y), \rho_V(y)) = \frac{1}{Z(\rho_R(y), \rho_V(y))}\frac{\rho_R(y)^k}{k!}\rho_V(y)^l \tag{3}$$

where $Z(\rho_R(y), \rho_V(y))$ is a normalization constant. In the following, we will use both $\pi(\rho_R(y), \rho_V(y))$ and $\pi(y)$. Since $\pi(y)\in\mathcal{Y}$, we also have

$$s = \sum_{(k,l)\in\chi} (k+l)\pi_{k,l}(y). \tag{4}$$

It can be rewritten $s = \mathbb{E}(R+V)$ where (R,V) is a random variable with distribution $\pi(y)$.

For the steady-state behavior of the model, we are interested by the equilibrium points $\overline{y}$ of the ODE (1). Such equilibrium points $\overline{y}$ satisfies

$$\pi(\rho_R(y), \rho_V(y)) = y \tag{5}$$

and equations (2) and (4). Thus, this is equivalent to find couples (ρ_R, ρ_V) such that $\bar{y} = \pi(\rho_R, \rho_V)$, satisfying

$$\rho_R = \frac{\lambda}{\mu}(1-\pi_{\cdot,0}(\rho_R,\rho_V)) \;,\;\; \rho_V = \frac{1-\pi_{\cdot,0}(\rho_R,\rho_V)}{1-\pi_S(\rho_R,\rho_V)} \tag{6}$$

$$s = \mathbb{E}(R+V) \tag{7}$$

where (R,V) is a random variable with distribution $\pi(\rho_R, \rho_V)$. Hence, the question of finding a measure on χ is reduced to find a couple of real numbers (ρ_R, ρ_V).

3.2 Existence and Uniqueness of the Equilibrium Point

Our second main result is the existence, uniqueness and characterization of the equilibrium point.

THEOREM 2 (A UNIQUE EQUILIBRIUM POINT). *There is a unique equilibrium point $\bar{y}$ for the solution of ODE (1) given by $\bar{y} = \pi(\rho_R, \rho_V)$ where π is the invariant measure of a tandem of two queues with total capacity K and respective rates ρ_R and ρ_V such that (ρ_R, ρ_V) is the unique solution of the first equation of (6) and equation (7).*

PROOF. We present a sketch of the proof (see [2, Theorem 2] for a full proof). Recall that finding an equilibrium point y of the ODE (1) is equivalent to finding (ρ_R, ρ_V) solution of equations (6) and (7). First, the second equation in (6) is true for any (ρ_R, ρ_V). Thus (6) amounts to its first equation. Second, this first equation of (6) is an implicit equation in ρ_R and ρ_V. It gives, by implicit function theorem, a diffeomorphism $\phi : [0, \lambda/\mu[\to [0, +\infty[$ such that $\rho_V = \phi(\rho_R)$. Moreover we prove that ϕ is strictly increasing. The third step consists in a combinatorial proof of the monotonicity of the right-hand side of equation (7) as a function of ρ_V and ρ_R. Using the continuity and monotonicity of ϕ related ρ_R and ρ_V, we conclude to the monotonicity of the right-hand side of equation (7) as a function of ρ_R. It gives the existence and uniqueness of a ρ_R thus of (ρ_R, ρ_V), solution of (6) and (7). It ends the proof. $\square$

4. SYSTEM PERFORMANCE

The aim is to give insight into the optimal behavior for performance. The influence of fleet size parameter s is investigated. In [2] the authors discuss different performance metrics. Here we focus on the proportion of empty and full stations called *problematic* stations. Its asymptotics as the system and time gets large is a function of $\bar{y}$.

Definition 1 (Problematic Stations). *Let $\bar{y}$ be the unique equilibrium point of ODE* (1). *The stations with either no car or no parking space available are called problematic. The limiting stationary proportion P_b of problematic stations is given by*

$$P_b = \mathbb{P}(V = 0 \text{ or } R + V = K) = \bar{y}_{.,0} + \bar{y}_S - \bar{y}_{K,0}$$

where (R, V) is a random variable with distribution $\bar{y}$.

4.1 Influence of the fleet size

All functions of (ρ_R, ρ_V) can be expressed on ρ_V only, like $\bar{y}(\rho_V)$, $P_b(\rho_V)$ and $s(\rho_V)$. Indeed by first equation in (6), for a fixed $\rho_V \in [0, +\infty[$, $\rho_R = \phi^{-1}(\rho_V)$. Then we get $\bar{y} = \pi(\rho_R, \rho_V)$ by (3) and $s(\rho_V)$ by (4). Thus, with abuse of notation, the proportion of problematic stations, as a function of the fleet size, is given by the parametric curve

$$\rho_V \mapsto (s(\rho_V), P_b(\rho_V)).$$

This curve gives the influence of the fleet size on the behavior of the system. It is numerically plotted in Figure 2 and compared to the same curve for bike sharing systems studied in [3] as the performance metrics are the same.

Figure 2 shows a minimum for P_b as a function of s. We can characterize it by this result.

Proposition 1. *P_b has an extremum $P_b{}^*$ for $\rho_V = 1$.*

Proof. The following interesting property of symmetry can easily be checked. For $\rho_V > 0$, $\phi^{-1}(1/\rho_V) = \phi^{-1}(\rho_V)/\rho_V$ and $P_b(1/\rho_V) = P_b(\rho_V)$. Thus $P_b'(\rho_V) = -P_b'(1/\rho_V)/\rho_V^2$. Therefore, $P_b'(1) = -P_b'(1) = 0$. $\square$

Uniqueness of the minimum should come from the convexity of the parametric curve $\rho_V \mapsto (s(\rho_V), P_b(\rho_V))$. Nevertheless this convexity remains to prove, the implicit relation between ρ_V and ρ_R making calculations tedious. This fact is similar to the result in [3, Theorem 1] for bike-sharing systems, that the minimum is reached for $\rho_V = 1$.

4.2 Optimal fleet size

We can compute asymptotics for P_b and s for any ρ_V in two cases: light ($\lambda/\mu \to 0$) and heavy ($\lambda/\mu \to +\infty$) traffic (see [2] for details). Then taking $\rho_V = 1$ gives expansions in λ/μ for the optimal value P_b^* for s^*. The first terms of the expansions are presented in Table 1.

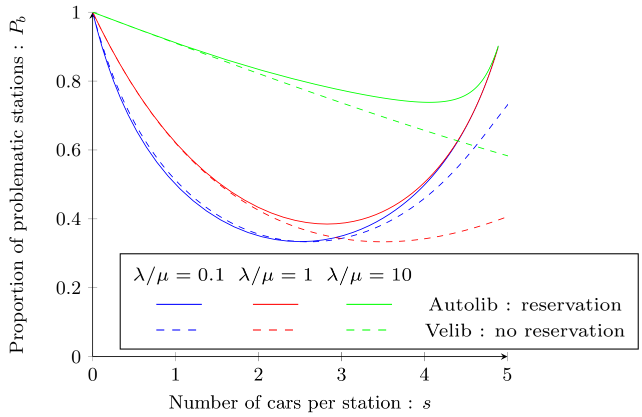

Figure 2: Asymptotic stationary proportion of empty and full stations as a function of the fleet size s for station capacity $K = 5$ (parametric curve).

	light traffic: $\lambda/\mu \to 0$	heavy traffic: $\lambda/\mu \to +\infty$
s^*	$\frac{K}{2} + \frac{K^2}{2(K+1)^2}\frac{\lambda}{\mu}$	$K - \sqrt{\frac{K}{\lambda/\mu}} + \frac{2-K}{\lambda/\mu}$
P_b^*	$\frac{2}{K+1} + \frac{2K}{(K+1)^3}\frac{\lambda}{\mu}$	$1 - \frac{K-1}{\lambda/\mu} + \frac{2(K-1)}{\sqrt{K}}(\frac{\lambda}{\mu})^{-3/2}$

Table 1: First terms for expansions in λ/μ at 0 (light traffic) and $+\infty$ (heavy traffic) of optimal proportion of problematic stations P_b^* for s^* cars per station.

For light traffic, the system has the same optimal performance at first order in λ/μ as the homogeneous bike-sharing model, where it is $2/(K+1)$ for each λ/μ (see [3, Theorem 1]). Indeed, intuitively, reservation does not induce congestion in light traffic case. This result can be observed on Figure 2 where curves for the bike-sharing and car-sharing systems are quite close at optimum only for $\lambda/\mu = 0.1$. In case of heavy traffic, P_b^* tends to 1 quicker compared to the convergence of s^* to K.

5. REFERENCES

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